SHARP INEQUALITIES FOR DETERMINANTS OF TOEPLITZ OPERATORS AND $\bar{\partial}-$LAPLACIANS ON LINE BUNDLES

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Abstract. We prove sharp inequalities for determinants of Toeplitz operators and twisted $\bar{\partial}-$Laplace operators on the two-sphere, generalizing the Moser-Trudinger-Onofri inequality. In particular a sharp version of conjectures of Gillet-Soulé and Fang motivated by Arakelov geometry is obtained; applications to $SU(2)-$invariant determinantal random point processes on the two-sphere are also discussed. The inequalities are obtained as corollaries of a general theorem about the maximizers of a certain non-local functional defined on the space of all positively curved Hermitian metrics on an ample line bundle L over a compact complex manifold. This functional is an "adjoint version", introduced by Berndtsson, of Donaldson's L-functional and generalizes the Ding-Tian functional whose critical points are Kähler-Einstein metrics. In particular, new proofs of some results in Kähler geometry are also obtained, including a lower bound on Mabuchi's $K-$energy and the uniqueness result for Kähler-Einstein metrics on Fano manifolds of Bando-Mabuchi.

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1. Introduction

Consider the two-dimensional sphere S^2 equipped with its standard Riemannian metric g_0 of constant positive curvature, normalized so that the corresponding volume form ω_0 gives unit volume to S^2. A celebrated inequality of Moser-Trudinger-Onofri proved in its sharp form by Onofri [41], asserts that

$$\log \int_{S^2} e^{-u}\omega_0 \leq -\int_{S^2} u\omega_0 + \frac{1}{4}\int_{S^2} du \wedge d^c u \tag{1.1}$$

for any, say smooth, function u on S^2, where the last term is the L^2-norm of the gradient of u in the conformally invariant notation of section 1.1 below.

As is well-known the inequality above has a rich geometric content and appears in a number of seemingly unrelated contexts ranging from the problem of prescribing the Gauss curvature in a conformal class of metrics on S^2 (the Yamabe and Nirenberg problems [18]) to sharp critical *Sobolev inequalites* [5] and lower bounds on *free energy functionals* in mathematical physics [41, 46]. The geometric content of the inequality above appears clearly when considering the extremal funtions u. Note first that $e^{-u}\omega_0$ appearing in the left hand side above is the volume form corresponding to the metric $g_u := e^{-u}g_0$, conformally equivalent to g_0. Denoting by $\mathrm{Conf}_0(g_0)$ the set of all metrics g_u with normalized volume (equal to one), *equality* holds in 1.1 for u such that $g_u \in \mathrm{Conf}_0(g_0)$ precisely when g_u is the pull-back of g_0 under a conformal transformation of S^2. Since, g_0 has constant curvature, this latter fact means that u satisfies the constant positive curvature equation

$$\omega_0 + dd^c u = e^{-u}\omega_0,$$

where $dd^c u$ is proportional to $(\Delta_{g_0}u)\omega_0$ (using the notation in section 1.1).

There is also a *spectral* intepretation of the Moser-Trudinger-Onofri inequality. As shown by Onofri [41] and Osgood-Phillips-Sarnak [42] the inequality 1.1 is equivalent to the fact that the functional

$$g_u \mapsto \det \Delta_{g_u}$$

on $\mathrm{Conf}_0(g_0)$, where $\det \Delta_{g_u}$ denotes the (zeta function regularized) *determinant of the Laplacian* Δ_{g_u} wrt the metric g_u, achieves its upper bound precisely for g_0 (modulo conformal transformations as above). The bridge between this latter fact and the inequality 1.1 is given by the *Polyakov anomaly formula* [18], which first appeared in Physics in the path integral (random surface) approach to the quantization of the bosonic string.

From the point of view of complex geometry (S^2, g_0) may be identified with the complex projective line $\mathbb{P}^1$ endowed with its standard $SU(2)-$invariant Kähler metric ω_0 (the Fubini-Study metric). The two-form ω_0 is the normalized curvature form of an Hermitian metric h_0 on the hyper plane line bundle $\mathcal{O}(1) \to \mathbb{P}^1$. For any natural number m, the pair (ω_0, h_0) induces naturally

- an $SU(2)-$invariant Hermitian product on the space $H^0(\mathbb{P}^1, \mathcal{O}(m))$ of global holomorphic sections of the m th tensor power $\mathcal{O}(m)$ of $\mathcal{O}(1)$, i.e. on the space of all homogenous polynomials of degree m.
- a *Dolbeault Laplace operator* $\Delta^{(m)}_{\bar{\partial}_0}$ (i.e. a $\bar{\partial}-$Laplacian) on the space of smooth sections of $\mathcal{O}(m)$, such that its null-space is precisely $H^0(\mathbb{P}^1, \mathcal{O}(m))$.

Changing the Hermitian metric on $\mathcal{O}(m)$ corresponds to "twisting" by a function e^{-u}, for $u \in \mathcal{C}^{\infty}(S^2)$ and we will denote the corresponding Dolbeault Laplace operator by $\Delta_{\bar{\partial}_u}^{(m)}$ (see section 1.2.1 for precise definitions). It is worth emphasizing that as opposed to the Laplacian Δ_{g_u} the Dolbeault Laplacian $\Delta_{\bar{\partial}_u}^{(m)}$ is *invariant* under translation of u. Motivated by Arakelov Geometry - notably the arithmetic Riemann-Roch theorem - Gillet-Soulé made a general conjecture which in the case of S^2 amounts to the following ([32]; see also [31] p. 526-527)

Conjecture. *(Gillet-Soulé). The determinant of the Dolbeault Laplacian $\Delta_{\bar{\partial}_u}^{(m)}$ naturally induced by the function u on any given line bundle $\mathcal{O}(m)$ over S^2 is bounded from above when u ranges over $\mathcal{C}^{\infty}(S^2)$.*

This was confirmed by Fang [30], who by symmetrization reduced the problem to the case when u is invariant under rotation around an axes of S^2, earlier treated by Gilllet-Soulé [32]. Fang also put forward the following more precise form of the conjecture above:

Conjecture. *(Fang). The upper bound in the previous conjecture is achieved precisely for u identically constant.*

As pointed out by Fang one motivation for this latter conjecture is that, after introducing suitable numerical constants depending on m in the right hand side of 1.1, it is implied be an inequality whose formulation is obtained by replacing $\int_X e^{-u}\omega_0$ by the determinant of the *Toeplitz operator with symbol e^{-u}* acting on the space $H^0(\mathbb{P}^1, \mathcal{O}(m))$. In this latter form the conjecture can be seen as a holomorphic analogue of an inequality appearing in connection to the classical *Szegö strong limit theorem* on S^1 (see chapter 3.1 in [34]). The relation between inequalities of Toeplitz operators on the sphere and upper bounds on determinants of Dolbeault Laplacians is a direct consequence of the anomaly formula of Bismut-Gillet-Soulé [14], which generalizes Polyakov's formula referred to above. It should also be pointed out that Toeplitz operators appear naturally in the *Berezin-Toeplitz quantization* of Kähler manifolds and in *microlocal analysis* [1].

In this paper the positive solution of Fang's conjecture will be deduced from a general result about the maximizers of a non-local functional $\mathcal{F}_{\omega_0}$ defined on the space of all positively curved Hermitian metrics on an ample line bundle L over a Kähler manifold (X, ω_0). In fact, a more precise inequality then the one conjectured by Fang will be obtained (Corollary 3) which implies both Fang's conjecture *and* the Moser-Trudinger-Onofri inequality above (and hence the extremal properties of $\det \Delta_{g_u}$, as well). The inequality obtained is equivalent to the upper bound

$$\log\left(\frac{\det \Delta_{\bar{\partial}_u}^{(m)}}{\det \Delta_{\bar{\partial}_0}^{(m)}}\right) \leq -\frac{1}{2}\left(\frac{1}{(m+2)}\right)\int du \wedge d^c u (\leq 0), \tag{1.2}$$

where $\det \Delta^{(m)}_{\overline{\partial}_u}$ is the Dolbeault Laplacian corresponding to $\mathcal{O}(m)$, which clearly implies Fang's conjecture above. The extremals in the first inequality above will also be characterized.

As pointed out in remark 14 the inequality in Corollary 3 is *sharp* in a rather strong sense. Moreover, in the limit when m tends to infinity, while the function u is kept fixed, the inequality becomes an asymptotic *equality.* As it turns out, this latter fact is essentially equivalent to a Central Limit Theorem for a certain random point process on the sphere. This process appears naturally as a *random matrix model* and as a *one component plasma* in the statistical physics litterature (see section 4).

The functional $\mathcal{F}_{\omega_0}$ referred to above is an "adjoint version", introduced by Berndtsson, of Donaldson's (normalized) L-functional and generalizes the Ding-Tian functional whose critical points are Kähler-Einstien metrics. In particular, new proofs of some results in Kähler geometry are also obtained, including a lower bound on Mabuchi's $K-$energy and the uniqueness result for Kähler-Einstein metrics on Fano manifolds of Bando-Mabuchi (see section 1.3 for precise references).

The relation between the inequality 1.1 and Kähler-Einstein metrics of positive curvature in higher dimensions seems to first have been suggested by Aubin [3]. It should also be pointed out that recently Rubinstein [40, 39] gave a different complex geometric proof of the inequality 1.1 using the inverse Ricci operator and its relation to various energy functionals in Kähler geometry. See also Müller-Wendland [43] for a proof of the result on extremals of determinants of the scalar Laplacian using the Ricci flow. However, these latter methods seem to be less well adapted to the non-local variational equations which appear in the setting of Gilllet-Soulé's and Fang's conjectures.

Before turning to the precise statement of the main result we will first introduce the general setup.

1.1. **Setup.** Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X of complex dimension n. Denote by $\mathrm{Aut}_0(X, L)$ the group of automorphism of (X, L) in the connected component of the identity, modulo the elements that act as the identity on X. The line bundle L will be assumed *ample,* i.e. there exists a Kähler form ω_0 in the first Chern class $c_1(L)$ and a "weight" ψ_0 on L such that ω_0 is the normalized curvature $(1,1)-$form of the hermitian metric on L locally represented as $h_0 = e^{-\psi_0}$. In this notation, the space of all positively curved smooth hermitian metrics on L may be identified with the open convex subset

$$\mathcal{H}_{\omega_0} := \{u : \omega_u := dd^c u + \omega_0 > 0\}$$

of $\mathcal{C}^\infty(X)$, where $d^c := i(-\partial + \overline{\partial})/4\pi$, so that $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. Note that, under this identification, the natural action of $\mathrm{Aut}_0(X, L)$ on the space of all metrics on L corresponds to the action $(u, F) \mapsto v := F^*(\psi_0 + u) - \psi_0$ so that, in particular, $\omega_v = F^*\omega_u$. Occasionally, we will also work with the closure $\overline{\mathcal{H}}_{\omega_0}$ of $\mathcal{H}_{\omega_0}$ in $L^1(X, \omega_0)$, coinciding with the space of all

ω_0-plurisubharmonic functions on X, i.e. the space of all upper semi-continuous functions u which are absolutely integrable and such that $\omega_u \geq 0$ as a $(1,1)-$current.

We equip the $N-$dimensional complex vector space $H^0(X, L+K_X)$ of all holomorphic sections of the adjoint bundle $L+K_X$ where K_X is the canonical line bundle on X, with the Hermitian product induced by ψ_0, i.e.

$$\langle s, s \rangle_{\psi_0} := i^{n^2} \int_X s \wedge \bar{s} e^{-\psi_0},$$

identifying s with a holomorphic $n-$form with values in L. We will use additive notation for tensor products of line bundles.

1.2. **Statement of the main results.** Next, we will introduce the two functionals on $\mathcal{H}_{\omega_0}$ which will play a leading role in the following. First, consider the following energy functional

$$\mathcal{E}_{\omega_0}(u) := \frac{1}{(n+1)!V} \sum_{i=1}^{n} \int_X u(dd^c u + \omega_0)^j \wedge (\omega_0)^{n-j}, \tag{1.3}$$

where $V := \mathrm{Vol}(\omega_0)$ is the volume of L, which seems to first have appeared in the work of Mabuchi [44] and Aubin [3]in Kähler geometry ($\mathcal{E}_{\omega_0} = -F^0_{\omega_0}$ in the notation of [51]). It also appears in Arithmetic (Arakelov) geometry as the top degree component of the secondary Bott-Chern class of L attached to the Chern character.

The second functional $\mathcal{L}_{\omega_0}$ may be geometrically defined as $\frac{1}{N}$ times the logarithm of the quotient of the volumes of the unit-balls in $H^0(X, L+K_X)$ defined by the Hermitian products induced by the metrics ψ_0 and $\psi_0 + u$ [9]. Concretely, this means that

$$\mathcal{L}_{\omega_0}(u) := -\frac{1}{N} \log\det(\langle s_i, s_j \rangle_{\psi_0+u}), \tag{1.4}$$

where $1 \leq i, j \leq N$ and s_i is any given base in $H^0(X, L+K_X)$ which is orthogonal wrt $\langle \cdot_i, \cdot \rangle_{\psi_0}$. The functional $\mathcal{L}_{\omega_0}(u)$ may also be invariantly expressed as a *Toeplitz determinant:*

$$\mathcal{L}_{\omega_0}(u) := -\frac{1}{N} \log\det(T[e^{-u}]), \tag{1.5}$$

where $T[e^{-u}]$ is the *Toeplitz operator with symbol* e^{-u} defined as the linear operator $\Pi_L \circ e^{-u}\cdot$ on $H^0(X, L+K_X)$, expressed in terms of the the orthogonal projection $\Pi_L : \mathcal{C}^\infty(X) \to H^0(X, L+K_X)$ (compare formula 6.2 in the appendix). If $N = 0$ we let $\mathcal{L}_{\omega_0}(u) := -\infty$. The normalizations are made so that the functional

$$\mathcal{F}_{\omega_0} := \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$$

is invariant under addition of constants and hence descends to a functional on the space of all Kähler metrics in $c_1(L)$. An element u in $\mathcal{H}_{\omega_0}$ will be said to be *critical* (wrt $L+K_X$) if it is a critical point of the

functional $\mathcal{F}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$, i.e. if u is a smooth solution in $\mathcal{H}_{\omega_0}$ of the Euler-Lagrange equations $(d\mathcal{F}_{\omega_0})_u = 0$. These equations may be written as the highly non-linear Monge-Ampère equation:

$$\frac{1}{Vn!}(dd^c u + \omega_0)^n = \beta(u), \tag{1.6}$$

where $\beta(u)$ is the Bergman measure associated to u (formula 2.4 below). This latter measure depends on u in a *non-local* manner and is strictly positive precisely when $L + K_X$ is *globally generated,* i.e. when there, given any point x in X, exists an element s in $H^0(X, L + K_X)$ such that $s(x) \neq 0$. For example, since L is ample, this condition holds when L is replaced by kL for k sufficiently large.

By definition, a critical point u is a priori only a *local* extremum of $\mathcal{F}_{\omega_0}$. But the next theorem relates *global* maximizers of $\mathcal{F}_{\omega_0}$ and its critical points:

Theorem 1. Let L be an ample line bundle such that the adjoint line bundle $L + K_X$ is globally generated. Then the absolute maximum of the functional $\mathcal{F}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$ is attained at any critical point u. Moreover, any smooth maximizer of $\mathcal{F}_{\omega_0}$ on $\overline{\mathcal{H}}_{\omega_0}$ is unique (up to addition of constants) modulo the action of $Aut_0(X, L)$. In particular, such a maximizer is critical.

In the case when the ample line bundle $L = -K_X$, so that X is a Fano manifold, the space $H^0(X, L + K_X)$ is one-dimensional and hence $\mathcal{L}_{\omega_0}(u) = -\frac{1}{N}\log\int e^{-(u+\psi_0)}$. Then it is well-known that any critical point may be identified with a Kähler-Einstein metric on X.

It should be emphasized that the *existence* of critical points of $\mathcal{F}_{\omega_0}$ is a very difficult issue closely related to conjectures of Yau, Tian, Donaldson and others in Kähler geometry [51, 27, 49]. Even in the case $L = -K_X$ there are well-known examples already on complex surfaces, where critical points do not exist.

Next, assume that (X, L) is $K-$*homogenous,* i.e. that X admits a transitive action by a compact semi-simple Lie group K, whose action on X lifts to L. We will then take ω_0 as the unique Kähler form in $c_1(L)$ which is invariant under the action of K on X.

Corollary 2. *Let $L \to X$ be a $K-$homogenous ample holomorphic line bundle over a compact complex manifold X and denote by ω_0 be the unique $K-$invariant Kähler metric in $c_1(L)$. Then, for any function u in $\mathcal{H}_{\omega_0}$*

$$-\mathcal{L}_{\omega_0}(u) \leq -\mathcal{E}_{\omega_0}(u)$$

with equality iff the function u is constant, modulo the action of $Aut_0(X, L)$.

Surprisingly, specializing to the case when X is a complex curve (i.e. $n = 1$) allows one to take u as *any* smooth function (which is *not* true in higher dimensions, as shown in [39] in the case when $L = -K_X$; see remark 12). More generally, we can then take u to be in the Sobolev

space $W^{2,1}(X)$ of all functions u on X such that u and its differential du are square integrable.

We will next consider the homegenous case, i.e. when $X = \mathbb{P}^1$, the complex projective line (i.e. topologically $X = S^2$, the two-sphere) and hence $K = SU(2)$. In this case any ample line bundle L may be written as $\mathcal{O}(k)$, where k is a positive integer and $H^0(\mathbb{P}^1, \mathcal{O}(k) + K_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}(k-2))$ may be identified with the space of all polynomials of at most degree $m := k-2$ on the affine piece $\mathbb{C}$ in $\mathbb{P}^1$ (assuming $k \geq 2$). Moreover, if we take $\psi_{0,k}(z) = k \log(1 + z\bar{z})$ as the fixed invariant weight on $\mathcal{O}(k)$, in the usual trivialization over the affine piece $\mathbb{C}$, then, under the identification above, the Hermitian product on $H^0(\mathbb{P}^1, \mathcal{O}(k) + K_{\mathbb{P}^1})$ may be written as

$$\langle p_m, p_m \rangle_{\psi_0 + u} := \int_{\mathbb{C}} \frac{|p_m|^2}{(1 + z\bar{z})^m} e^{-u} \omega_0 \tag{1.7}$$

for $p_m(z)$ a polynomial on $\mathbb{C}$ of degree at most m (compare section 3.3.1). Hence,

$$\mathcal{L}_{\omega_{0,k}}(u) := (m+1)\mathcal{L}_m(u) := -\log\det(c_{ij} \int_{\mathbb{C}} \frac{z^i \bar{z}^j}{(1 + z\bar{z})^m} e^{-u} \omega_0),$$

where $i, j = 0, ..., m$ and $1/c_{ij} = (m+1)\binom{m}{i}\binom{m}{j}$.

Corollary 3. *Let u be a function in the Sobolev space $W^{2,1}(S^2)$ on the two-sphere S^2 and denote by ω_0 the volume form corresponding to the metric on S^2 with constant curvature and volume one. Then*

$$-\mathcal{L}_m(u) \leq -(m+1) \int_{S^2} u\omega_0 + (\frac{m+1}{m+2}) \frac{1}{2} \int_{S^2} du \wedge du^c$$

with equality iff there exists a Möbius transformation M of S^2 such that $\omega_u = M^* \omega_0$.

The case when $m = 0$, so that $-\mathcal{L}_m(u) = \log \int_{\mathbb{C}} e^{-u}\omega_0$, is precisely the celebrated Moser-Trudinger-Onofri inequality 1.1. The reduction of the proof of Corollary 3 to Corollary 2, is based on properties of the projection operator P_ω (formula 1.8).

1.2.1. *Application to determinants of $\bar{\partial}-$Laplace operators and Analytic Torsion.* Consider again the case when $X = \mathbb{P}^1$ is the complex projective line equipped with the standard Kähler form ω_0. Any function u corresponds to a metric on $\mathcal{O}(m)$ with weight $m\psi_0 + u$, where m is a fixed non-negative integer. Hence, the pair (ω_0, u) induces natural Hilbert norms on the space $\Omega^{0,q}(\mathcal{O}(m))$ of smooth $(0, q)-$forms with values in $\mathcal{O}(m)$, where $q = 0, 1$. Denote by $\Delta^{(m)}_{\bar{\partial}_u}$ the corresponding $\bar{\partial}-$Laplace (Dolbeault) operator acting on the space $\Omega^0(\mathcal{O}(m)$, i.e. $\Delta^{(m)}_{\bar{\partial}_u} = \bar{\partial}^* \bar{\partial}$, where $\bar{\partial}^*$ is the formal adjoint of the $\bar{\partial}-$operator

$$\bar{\partial} : \ \Omega^{0,0}(\mathcal{O}(m)) \rightarrow \Omega^{0,1}(\mathcal{O}(m))$$

Note that $\bar{\partial}^*$ may be expressed in terms of the adjoint $\bar{\partial}^{*,0}$ induced by $u=0$ as

$$\bar{\partial}^* = e^u \bar{\partial}^{*,0} e^{-u}$$

The zeta function regularized *determinant* of the operator obtained by restricting $\Delta_{\bar{\partial}_u}^{(m)}$ to the orthogonal complement of its kernel will be denoted by $\det \Delta_{\bar{\partial}_u}^{(m)}$ (compare [14]). Given the result in the previous corollary, the anomaly formula (i.e. a family Riemann-Roch-Grothendieck theorem) of Bismut-Gillet-Soulé [14] now implies the following positive solution of Fang's conjecture

Corollary 4. *Given the line bundle $\mathcal{O}(m) \to \mathbb{P}^1$, the corresponding functional*

$$u \mapsto \det \Delta_{\bar{\partial}_u}^{(m)}$$

on the space of all smooth functions u on $\mathbb{P}^1$ attains its maximum precisely for u a constant function.

In fact, the proof of the previous Corollary, will give the stronger statement that the inequality 1.2 for $\det \Delta_{\bar{\partial}_u}$ stated in the introduction holds and that this latter inequality is equivalent to Corollary 3. Note that a direct consequence of the previous corollary is the following reponse to a variant of Kac's classical question "Can one hear the shape of a drum?" [37]: if the $\bar{\partial}-$Laplacian on *some* power $\mathcal{O}(m)$ induced by a smooth metric h on $\mathcal{O}(1) \to \mathbb{P}^1$ has the same spectrum (including multiplicities) as the $\bar{\partial}-$Laplacian induced by the standard $SU(2)$ invariant metric h_0, then $h = Ch_0$ for a positive number C.

Finally it should be pointed out that in the general case of an ample line bundle L Theorem 1 yields a bound on the twisted *Ray-Singer analytic torsion* (see for example [14]) associated to a semi-positively curved metric on L in terms of the corresponding *Quillen metric* and the functional $\mathcal{E}$. This is a direct consquence of the fact that $L + K_X$ is ample, so that the higher cohomology groups $H^q(X, L + K_X)$, $q \geq 1$, vanish, combined with the anomaly formula of Bismut-Gillet-Soulé [14]. For the sake of brevity the details are omitted.

1.3. **Further relations to previous results.** In this case when $L = -K_X$ the first statement of Theorem 1 is a result of Ding-Tian[25] and the "uniqueness" of critical points (i.e. Kähler-Einstein metrics in this case) was proved earlier by Bando-Mabuchi [4]. See [10] for a generalization of this latter result to funtions of "finite energy", in the case when $\mathrm{Aut}_0(X, L)$ is discrete (compare remark 5).

The extremal property of the critical points in Theorem 1 can also be seen as an analog of a result of Donaldson (Theorem 2 in [28]) who furthermore assumed that $\mathrm{Aut}_0(X, L)$ is discrete. In this latter setting the role of the space $H^0(X, L+K_X)$ is played by $H^0(X, L)$ equipped with the scalar products induced by the weight $\psi_0 + u$ and the integration measure $(\omega_u)^n/n!$ Note however that in Donaldson's setting the functional corresponding to $\mathcal{F}_{\omega_0}$ is *minimized* on its critical points (compare section 5.1

and the discussion in section 5 in [13]). In the terminilogy of [28] these latter critical points correpond to *balanced metrics.* Donaldson used his result, combined with the deep convergence results in [27] for balanced metrics, in the limit when L is replaced by a large tensor power, to prove a lower bound on Mabuchi's K-energy functional. It will be shown in section 5 how to deduce this latter result more directly from Theorem 1 above.

It should also be pointed out that the inequality proved by Donaldson corresponds to a *lower* bound on $\mathcal{F}_{\omega_0}(u)$ in the present setting, which however will depend on u through its volume form $(\omega_u)^n/n!$ (see the end of section5.1).

1.4. **Concerning the proof of Theorem 1.** The proof of Theorem 1 relies on the recent work [13] of Berndtsson combined with some global pluripotential theory developed in [9, 11] (see also [10] for the case $L = -K_X$). On one hand [13] gives that $\mathcal{F}_{\omega_0}$ is "geodesically" convex wrt the Riemann metric on the space $\mathcal{H}_{\omega_0}$ introduced by Mabuchi [43]. In turn, this fact is used to show that any critical point maximizes $\mathcal{F}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$, using the existence of (generalized) C^0-geodesics in the closure $\overline{\mathcal{H}}_{\omega_0}$. On the other hand, a main point in the proof of the "uniqueness" of critical points is to show that there are no smooth extremal points of $\mathcal{F}_{\omega_0}$ in the "boundary" of $\mathcal{H}_{\omega_0}$, i.e. in $\overline{\mathcal{H}}_{\omega_0} - \mathcal{H}_{\omega_0}$. Following [9, 10] this is shown by extending $\mathcal{F}_{\omega_0}$ to a (Gâteaux) differentiable function on all of $C^0(X)$, by replacing $\mathcal{E}_{\omega_0}$ with the composed map $\mathcal{E}_{\omega_0} \circ P_{\omega_0}$, where P_{ω_0} is the following (non-linear) projection operator from $C^0(X)$ onto $\mathcal{C}^0(X) \cap \overline{\mathcal{H}}_{\omega_0}$:

$$P_{\omega_0}[u](x) = \sup \{v(x) : \ v \in \mathcal{H}_{\omega_0}, \ v \leq u\} \tag{1.8}$$

Remark 5. Consider the setting of Theorem 1 and assume that there exists a (smooth) critical point, which we may assume is given by 0. Then the inequality furnised by the theorem, i.e.

$$\mathcal{F}_{\omega_0}(u) := \mathcal{E}_{\omega_0}(u) - \mathcal{L}_{\omega_0}(u) \leq 0$$

actually holds for all u in $\mathcal{E}^1(X, \omega_0)$, i.e. for al u in the convex set of all u in $\overline{\mathcal{H}}_{\omega_0}$ with *finite energy;* $\mathcal{E}(u) > -\infty$, where

$$\mathcal{E}(u) := \inf_{u' \geq u} \mathcal{E}(u')$$

when u' ranges over all elements in $\mathcal{H}_{\omega_0}$ such that $u' \geq u$. Equivalently, $\int_X (\omega_u)^n = \mathrm{Vol}(L)$ and $-\int_X u(\omega_u)^n < \infty$ in terms of *non-pluripolar products* (see [10] and references therein). The inequality on all of $\mathcal{E}^1(X, \omega_0)$ is simply obtained by writing u as a decreasing limit of elements in $\mathcal{H}_{\omega_0}$ and using the continuity of $\mathcal{E}$ and $\mathcal{L}_{\omega_0}$ under such limits [10] (note that e^{-u} is integrable if $\mathcal{E}(u) > -\infty$ [10]).

Moreover, in the case when $\mathrm{Aut}_0(X, L)$ is discrete it can be shown that any maximizer of $\mathcal{F}_{\omega_0}$ on $\mathcal{E}^1(X, \omega_0)$, is in fact equal to a constant. The proof is a simple adaptation of the argument in [10] concerning the case $L = -K_X$. It would be interesting to know if the general uniqueness statement in Theorem 1 also remains true in the larger class $\mathcal{E}^1(X, \omega_0)$?

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Organization. In section 2 preliminaires for the proofs of the main results appearing in section 3 are given. The proof of the uniqueness statement in the main theorem relies on higher order regularity for "geodesics" defined by inhomogenous Monge-Ampère equations. An alternative proof based on considerably more elementary regularity results is given in section 3.6. In section 3.5 applications to Arithmetic (Arakelov) geometry are briefly indicated. In section 4 some of the previous results are interpreted in terms of $SU(2)-$invariant determinantal random point process on S^2. Finally, in section 5 the limit when the line bundle L is replaced by a large tensor power is studied and a new proof of the lower bound on Mabuchi's $K-$energy for a polarized projective manifold is given. Relations to Donaldson's work are also discussed. In the appendix some formulas involving Bergman kernels are recalled and a "Bergman kernel proof" of Theorem 9 is given.

2. Preliminaries: Geodesics and energy functionals

2.1. **Geodesics.** The infinite dimensional space $\mathcal{H}_\omega$ inherits an *affine* Riemannian structure from its natural imbedding as on open set in $\mathcal{C}^\infty(X)$. Mabuchi, Semmes and Donaldson (see [19] and references therein) introduced another Riemannian structure on $\mathcal{H}_\omega$ (modolo the constants) defined in the following way. Identifying the tangent space of $\mathcal{H}_\omega$ at the point u with $\mathcal{C}^\infty(X)$ the squared norm of a tangent vector v at the point u is defined as

$$\int_X v^2(\omega_u)^n/n!.$$

However, the *existence* of a geodesic u_t in $\mathcal{H}_\omega$ connecting any given points u_0 and u_1 is an open and even dubious problem. There are two problems: it is not known if $i)$ u_t smooth, $ii)$ ω_{u_t} is strictly positive, as a current. As is well-known such a geodesic may, if it exists, be obtained as the solution of a homogenous Monge-Ampère equation (see below). In the following we will simply take this characterization as the *definition* of a geodesic. It will also be important to consider the larger space $\overline{\mathcal{H}}_{\omega_0} \cap C^0(X)$, since a priori the path u_t may leave $\mathcal{H}_\omega$.

Definition 6. A continuous path in $\overline{\mathcal{H}}_{\omega_0} \cap C^0(X)$ u_t will be called a $\mathcal{C}^0-$*geodesic* connecting u_0 and u_1 if $U(w,x) := u_t(x)$, where $t = \log|w|$, is continuous on

$$M := \{1 \leq |w| \leq e\} \times X := A \times X$$

with $dd^cU + \pi_X^*\omega_0 \geq 0$ and

$$(dd^cU + \pi_X^*\omega_0)^{n+1} = 0 \tag{2.1}$$

in the interiour of M in the sense of pluripotential theory [33, 23], where π_X denotes the projection from M to X.

As shown in[11, 10] $U(w,x)$ exists and is uniquely defined as the extension from ∂M obtained as the upper envelope

$$U(w,x) = \sup\left\{V(w,x) : V \in \mathcal{H}_{\pi_X^*\omega_0}(M),\ V \leq U \text{ on } \partial M\right\}, \tag{2.2}$$

where $\mathcal{H}_{\pi_X^*\omega_0}(M)$ denotes the set of all smooth functions V on M such that $dd^cU + \pi_X^*\omega_0 > 0$. If u_t is such that $dd^cU + \pi_X^*\omega_0 \geq 0$ then u_t will be called a *psh path* (or a *subgeodesic*). In local computations we will often make the identification $u_t(x) = U(w,x)$ extending t to a complex variable. Then $u_t(x)$ is independent of the imaginary part of t and is hence *convex* wrt real t.

In the proof of the uniqueness part of Theorem 1 we will have great use for the following regularity result for geodesics in $\overline{\mathcal{H}}_{\omega_0}$, shown by Chen [19]. See also [16] for a detailed analysis of the proof and some refinements. The proof uses the method of continuity combined with very precise a prioiri estimates on the perturbed Monge-Ampère equations.

Theorem 7. *(Chen) Assume that the boundary data in the Dirichlet problem 2.1 for the Monge-Ampère operator on M is smooth on ∂M. Then $U \in \mathcal{C}^{1,1}_{\mathbb{C}}(M)$. More precisely, the mixed second order complex derivatives of U are uniformly bounded, i.e. there is a positive constant C such that*

$$0 \leq (dd^cU + \pi_X^*\omega_0) \leq C(\pi_X^*\omega_0 + \pi_A^*\omega_A)$$

where ω_A is the Eucledian metric on A.

In the statement above we have used the (non-standard) notation $\mathcal{C}^{1,1}_{\mathbb{C}}(M)$ for the set of all functions U such that, locally, the current dd^cU has coefficents in L^∞. Such a U is called *almost* $\mathcal{C}^{1,1}$ in [16]. Note that if $U \in \mathcal{H}_{\pi_X^*\omega_0}(M)$ then this is equivalent to U having a bounded Laplacian $\Delta_M U$, where Δ_M is the Laplacian on M wrt the Kähler metric $\pi_X^*\omega_0 + \pi_A^*\omega_A$ on M. As will be explained in section 3.6 the proof of the uniqueness statement in Theorem 1 may actually be obtained by only using the bounds on the derivatives of u_t on X for t fixed. As shown very recently in [11] such bounds may be obtained by working directly with the envelope 2.2.

Theorem 8. *Assume that the boundary data in the Dirichlet problem 2.1 for the Monge-Ampère operator on M is in $\mathcal{C}^{1,1}(\partial M)$. Then $u_t \in \mathcal{C}^{1,1}_{\mathbb{C}}(X)$. More precisely, the mixed second order complex derivatives of u_t on X are uniformly bounded, i.e. there is a positive constant C such that*

$$0 \leq (dd^cu_t + \omega_0) \leq C\omega_0$$

on X.

One of the virtues of this latter approach is that the proof is remarkly simple when X is homogenous.

2.2. **The functional $\mathcal{L}_{\omega_0}$.**. First note that the functional $\mathcal{L}_{\omega_0}(u)$ defined by formula 1.5 is increasing on $\mathcal{C}^0(X)$, wrt the usual order relation. This is an immediate consequence of the basic geometric interpretation in [9] of $\mathcal{L}_{\omega_0}(u)$ as propopertional to the logarithmic volume of the unit-ball in the Hilbert space $H^0(X, L+K_X)$ equipped with the Hermitian product induced by the weight $\psi_0 + u$. Alternatively, it follows from formula 2.3 below which shows that the differential of the functional $\mathcal{L}_{\omega_0}$ on $\mathcal{C}^0(X)$ may be represented by the positive measure β_u. Integrating β_u along a line segment in $\mathcal{C}^0(X)$ equipped with its affine structure then shows that $\mathcal{L}_{\omega_0}(u)$ is increasing.

The *differential* of the functional $\mathcal{L}_{\omega_0}$ on $\mathcal{C}^0(X)$ is given by

$$(d\mathcal{L}_{\omega_0})_u = \beta_u, \tag{2.3}$$

in the sense that given any smooth function v we have that

$$d(\mathcal{L}_{\omega_0}(u+tv))/dt_{t=0} = \int_X \beta_u v,$$

where β_u is the *Bergman measure associated to* u. This latter measure is the positive measure on X defined as

$$\beta_u = (i^{n^2} \frac{1}{N} \sum_{i=1}^{N} s_i \wedge \bar{s}_i e^{-\psi_0}) e^{-u} \tag{2.4}$$

in terms of any given orthonormal base (s_i) in the Hilbert space $H^0(X, L+K_X)$ equipped with the Hermitian product induced by the weight $\psi_0 + u$ (compare section 6.1). In particular this means that β_u may be represented as e^{-u} times a strictly positive smooth measure on X if $L+K_X$ is globally generated. The proof of formula 2.3 follows more or less directly from the definition (see [13] for a geometric argument).

The following theorem, which is direct consequence of a result of Berndtsson about the curvature of direct image bundles [13], considers the *second* derivatives of $\mathcal{L}_{\omega_0}$ along a psh path. As a courtesy to the reader a proof of the theorem, using Bergman kernels, is given in the appendix.

Theorem 9. *(Berndtsson) Let u_t be a continuous psh path in $\mathcal{H}_{\omega_0}$. Then the function $t \mapsto \mathcal{L}_{\omega_0}(u_t)$ is convex. Moreover, if $\mathcal{L}_{\omega_0}(u_t)$ is affine and u_t is a* smooth *psh path with $\omega_{u_t} > 0$ on X for all t, then there is an automorphism S_1 of (X, L), homotopic to the identity, such that $u_1 - u_0 = S_1^* \psi_0 - \psi_0$.*

The convexity statement in [13] assumed in fact that u_t be *smooth.* However, by uniform approximation the convexity statement above in fact holds for any *continuous* psh path in $C^0(X)$. Indeed, if u_t is such a

path, then there exists, for example by Richbergs's approximation theorem [22], a sequence U^j converging uniformly towards U on M such that $dd^cU^j+\pi^*\omega_0>0$. Applying the theorem above to each U^j and letting j tend to infinity then gives that $f(t):=\mathcal{L}_{\omega_0}(u_t)$ is a uniform limit of convex functions and hence convex, proving the claim.

However, for the uniqueness statement the argument in [13] seems to require that ω_{u_t} be reasonably smooth in (t,x). Moreover, the assumption that $\omega_t>0$ is crucial to be able to define the vector fields V_t that integrate to the automorphism S_1 (see formula 3.6).

2.3. **The functional $\mathcal{E}_{\omega_0}$.** First recall the following well-known formula for the differential of the energy functional $\mathcal{E}_{\omega_0}$ defined by formula 1.3:

$$(d\mathcal{E}_{\omega_0})_u=\omega_u^n/n! \tag{2.5}$$

The following generalization from [9] of the previous formula to the functional $\mathcal{E}_{\omega_0}\circ P_{\omega_0}$, where P_{ω_0} is the non-linear projection 1.8, will be crucial for the proof of Theorem 1:

Theorem 10. *The functional $\mathcal{E}_{\omega_0}\circ P_{\omega_0}$ is Gâteaux differentiable on $\mathcal{C}^0(X)$. Its differential at the* point u is represented by the measure $\omega^n_{P_{\omega_0}u}/n!$, i.e. *given $u,v\in\mathcal{C}^0(X)$ the function $\mathcal{E}_{\omega_0}P_{\omega_0}(u+tv)$ is differentiable on $\mathbb{R}_t$ and*

$$d\mathcal{E}_{\omega_0}P_{\omega_0}(u+tv)/dt_{t=0}=\int_X v\omega^n_{P_{\omega_0}u}/n! \tag{2.6}$$

As for the second derivatives of $\mathcal{E}_{\omega_0}$ we have the following Proposition which is well-known (at least in the smooth case):

Proposition 11. *The following properties of $\mathcal{E}_{\omega_0}$ hold:*

- The functional $\mathcal{E}_{\omega_0}$ on $\overline{\mathcal{H}}_{\omega_0}\cap\mathcal{C}^0(X)$ is concave wrt the affine structure on $\mathcal{C}^0(X)$.
- Let u_t be a $\mathcal{C}^0$- geodesic in $\overline{\mathcal{H}}_{\omega_0}$ connecting u_0 and u_1. Then the functional $t\mapsto\mathcal{E}_{\omega_0}(u_t)$ is affine and continuous on $[0,1]$.

Proof. (A proof also appears in [10]). Recall the following well-known formula (see for example [9]):

$$d_td_t^c\mathcal{E}_{\omega_0}(u_t)=t_*(dd^cU+\pi^*\omega_0)^{n+1}/(n+1)!, \tag{2.7}$$

where t_* denotes the natural push-forward map from M to $\mathbb{C}_t$. In particular, setting $u_t=u_0+tu$ gives for real t $d^2\mathcal{E}_{\omega_0}(u_t)/d^2t=-\int_X|\partial u|^2\omega_0^n\leq 0$ (compare formula 3.25) which proves the first point of the proposition when u is smooth. To handle the general case one takes u_j in $\mathcal{H}_{\omega_0}$ converging uniformly to u and uses that, according to Bedford-Taylor's classical results, $\mathcal{E}_{\omega_0}$ is continuous under uniform limits in $\overline{\mathcal{H}}_{\omega_0}\cap\mathcal{C}^0(X)$ (see also [9]). This shows that $\mathcal{E}_{\omega_0}(u_t)$ is the limit of concave functions and hence concave. To prove the last point take a sequence U^j converging uniformly to U on M and such that $dd^cU^j+\pi^*\omega_0>0$ (compare the discussion below Theorem 9). By Bedford-Taylor $(dd^cU^j+\pi^*\omega_0)^{n+1}$ tends weakly to

$(dd^cU^j+\pi^*\omega_0)^{n+1}$ in the interiour of M. Hence, formula 2.7 shows that the second real derivatives of $\mathcal{E}_j(t) := \mathcal{E}(u_t^j)$ tend weakly to zero in the sense of distributions for $t \in]0,1[$. But since the sequence $\mathcal{E}_j(t)$ of smooth convex functions tends to $\mathcal{E}(t)$ it follows that $\mathcal{E}(t)$ is affine on $]0,1[$ and hence by continuity on all of $[0,1]$. To be more precise: since U is continuous on the compact set M the family u_t tends to u_0 and u_1 uniformly when $t \to 0$ and $t \to 1$, respectively. Finally, since $\mathcal{E}$ is continuous under uniform limits in $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^0(X)$ this proves that $\mathcal{E}$ is continuous up to the boundary on $[0,1]$. □

Before turning to the proof of Theorem 1, we recall the following basic cocycle property of the functional $\mathcal{F}_{\omega_0} := \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$:

$$\mathcal{F}_{\omega_{u_2}}(u_1) + \mathcal{F}_{\omega_{u_3}}(u_2) = \mathcal{F}_{\omega_{u_3}}(u_1), \tag{2.8}$$

which is a direct consequences of the corresponding cocyle properties of $\mathcal{E}_{\omega_0}$ and $\mathcal{L}_{\omega_0}$. These latter properties in turn are immediately obtained by integrating the corresponding differentials along line segments (compare [51]).

Remark 12. The funtional $\mathcal{E}_{\omega_0}$ may be expressed in terms of a generalized Dirichlet type energy J_{ω_0} :

$$-\mathcal{E}_{\omega_0}(u) = J_{\omega_0}(u) - \frac{1}{V}\int u\omega_0,$$

where J_{ω_0} is Aubin's energy functional

$$J_{\omega_0}(u) := \frac{1}{V}\sum_{i=1}^{n-1}\frac{i+1}{n+1}\int du \wedge du^c \wedge (\omega_0)^i \wedge (\omega_u)^{n-1-i} \tag{2.9}$$

(compare [51] p. 58). Note that if $n=1$ then J_{ω_0} is non-negative for *any* u, while the natural condition to obtain non-negativity when $n>1$ is that $\omega_u \geq 0$. On the other hand as shown in [39] (lemma 2.1), there are examples of general smooth u with $J_{\omega_0} < 0$ for any manifold X of dimension $n>1$. As a direct consequence it was shown in [39], in the case $L=-K_X$, that any such fucntion u violates the inequality in Theorem 1. A similar argument applies to a homogeneous line bundle L as in Corollary 2. Indeed, without affecting the value of $J_{\omega_0}(u)$ we may assume that $\int_X u\omega_0^n = 0$ so that $-\mathcal{E}_{\omega_0}(u) = J_{\omega_0}(u) < 0$. Now, using the notation of section 4 below,

$$-\mathcal{L}_{\omega_0}(u) = \log \mathbb{E}_N(e^{-(u(x_1)+...+u(x_n))}) \geq -\mathbb{E}_N(u(x_1)+...+u(x_n)),$$

using Jensen's inequality in the last step. Moreover, by formula 6.4 $\mathbb{E}_N(u(x_1)+...+u(x_n)) = \int_X u\beta_u)$. Since $\beta_u = \omega_0^n/V$ in the homogenous case (compare the proof of Corollary 2), this means that $-\mathcal{L}_{\omega_0}(u) \geq 0$. Hence, u violates the inequality referred to above.

3. Proofs of the main results

3.1. **Proof of Theorem 1.** By the cocycle property of $\mathcal{F}_{\omega_0}$ (see [10, 9]; it is shown by integrating the differential of $\mathcal{F}_{\omega_0}$ along line segments) we may without loss of generality assume that $u=0$ is critical. Take a *continuous* element u_1 in $\mathcal{H}_{\omega_0}$ and the corresponding $\mathcal{C}^0-$geodesic u_t connecting $u_0=0$ and u_1. Since u_t is a continuous path, combining Theorem 9 and Proposition 11 gives that $\mathcal{F}_{\omega_0}(t):=\mathcal{F}_{\omega_0}(u_t)$ is a continous concave function on $[0,1]$. Hence, the inequality in Theorem 1 will follow once we have shown that

$$\frac{d}{dt}_{t=0+}\mathcal{F}(u_t)\leq 0. \tag{3.1}$$

Of course, if u_t were known to be a *smooth* path then this would be an immediate consequence of the assumption that u_0 is critical combined with the chain rule (which would even yield equality above). To prove 3.1 first observe that by the concavity in Prop 11

$$(\mathcal{E}_{\omega_0}(u_t)-\mathcal{E}_{\omega_0}(u_0))/t\leq\frac{1}{t}\int_X(u_t-u_0)(\omega_{u_0})^n/n!$$

Hence, the monotone convergence theorem applied to the sequence $(u_t-u_0)/t$ which decreases to the right derivative v_0 of u_t at $t=0$ (using that u_t is convex in t) gives

$$\frac{d}{dt}_{t=0+}\mathcal{E}_{\omega_0}(u_t)\leq\int_X v_0(\omega_{u_0})^n/n! \tag{3.2}$$

Hence,

$$\frac{d}{dt}_{t=0+}\mathcal{F}(u_t)\leq\int_X((\omega_{u_0})^n/n!-\beta_{u_u})v_0=0,$$

where we have also used the dominated convergence theorem to differentiate $\mathcal{L}_{\omega_0}(u_t)$ (compare [10, 9]). This finishes the proof of 3.1and hence the first statement in the theorem follows.

Uniqueness: Assume now that u_1 is a smooth maximizer of $\mathcal{F}_{\omega_0}$ on $\overline{\mathcal{H}}_{\omega_0}$ i.e. that $\mathcal{F}_{\omega_0}(u_1)=\mathcal{F}_{\omega_0}(u_0)$ by the previous step. Since $\mathcal{F}_{\omega_0}(t):=\mathcal{F}_{\omega_0}(u_t)$ is continuous and concave it follows that u_t maximizes $\mathcal{F}_{\omega_0}$ on $\overline{\mathcal{H}}_{\omega_0}\cap\mathcal{C}_{\mathbb{C}}{}^{1,1}(X)$ for all t. Next, we will show that u_t satisfies the Euler-Lagrange equation 1.6 for any fixed t (see [10] for similar arguments). To this end fix $t=t_0$ and set $u_{t_0}:=u$. Given a smooth function v on X consider the function $f(t):=\mathcal{E}_{\omega_0}(P_{\omega_0}(u+tv))-\mathcal{L}_{\omega_0}(u+tv)$ on $\mathbb{R}_t$. Since, the functional $\mathcal{L}_{\omega_0}$ is increasing on $\mathcal{C}^0(X)$ we have $f(t)\leq\mathcal{F}_{\omega_0}(P_{\omega_0}(u+tv))$. By assumption this means that the maximal value of the function $f(t)$ is attained for $t=0$ (also using that $P_{\omega_0}u=u$). In particular, since by Theorem 10 $f(t)$ is differentiable $df/dt=0$ at $t=0$ and Theorem 10 and formula 2.3 hence show that the Euler-Lagrange equation 1.6 holds (since it holds when tested on any smooth function v).

Next, we will prove that $U \in \mathcal{C}^{\infty}(\dot{M})$, where $\dot{M}$ denotes the interiour of M. By Theorem 7 U is in $\mathcal{C}^{1,1}_{\mathbb{C}}(M)$. Moreover, by the homogenous Monge-Ampère equation 2.1 and the Euler-Lagrange equation 1.6 we have

$$(dd^c(U+|w|^2)+\pi_X^*\omega_0)^{n+1}=i\beta_u\wedge dw\wedge d\bar{w}$$

Hence, the following equation holds locally on $\mathbb{C}^{n+1}$ (where we for simplicity have kept the notation U for the function obtained after subtracting a smooth and hence harmless function from U) :

$$\det(\partial_{\zeta_i}\partial_{\bar{\zeta}_j}U)=e^{-U}\rho, \tag{3.3}$$

where ρ is a positive smooth function, depending on U (compare the discussion below formula 2.4). In particular, $\det(\partial_{\zeta_i}\partial_{\bar{\zeta}_j}U)$ is locally in $\mathcal{C}^{1,1}_{\mathbb{C}}$. But then Theorem 2.5 in [15], which is a complex analog of a result of Trudinger for fully non-linear elliptic operators (compare Evans-Krylov theory), gives that U is locally in the Hölder space $\mathcal{C}^{2,\alpha}$ for some $\alpha>0$. Now the equation 3.3 shows that $\det(\partial_{\zeta_i}\partial_{\bar{\zeta}_j}U)$ is also in $\mathcal{C}^{2,\alpha}$. Finally, since we have hence shown that $U\in C^2$, standard theory of uniformly elliptic operators then allows us to boot strap using 3.3 and deduce that $U\in\mathcal{C}^{\infty}$ locally (see Theorem 2.2 in [16]). Note also that by the Euler-Lagrange equation 1.6 we have a uniform lower bound $\omega_{u_t}^n>\delta\omega_0^n$ (also using the lower bound in formula 6.5 in the appendix). Combining the previous lower bound with the upper bound $\omega_{u_t}\leq C\omega_0$ from Theorem 7 then shows that there is a positive constant C',independent of t, such that

$$1/C'\omega_0\leq\omega_{u_t}\leq C'\omega_0 \tag{3.4}$$

Since, by the above arguments $\mathcal{F}_{\omega_0}(u_t)$ and $\mathcal{E}_{\omega_0}(u_t)$ are both affine (and even constant) it follows that $\mathcal{L}_{\omega_0}(u_t)$ is affine. In case U were smooth *up to the boundary* of M applying Theorem 9 would hence prove the uniqueness statement in Theorem 1. To prove the general case we may without loss of generality assume that $u_t(x)$ is smooth on $[0,1[\times X$ (otherwise we just apply the same argument on $[1/2,1[$ and $]0,1/2]$). For any $\epsilon>0$ Theorem 9 (see Theorem 2.6 in [13]) furnishes a 1-parameter holomorphic family S_t in $\mathrm{Aut}_0(X,L)$ with $t\in[0,1-\epsilon]$ defined by the ordinary differential equation

$$\frac{dS_t(x(t))}{dt}=d_X(S(x(t))[V_t]_{x(t)} \tag{3.5}$$

with the iniatial data $S_0=I$ (the identity), where V_t is the vector field on X of type $(1,0)$ defined by the equation

$$\omega_{u_t}(V_t,\cdot)=\bar{\partial}_X(\partial_t u), \tag{3.6}$$

where $\bar{\partial}_X$ is the $\bar{\partial}-$operator on X and ∂_t is the partial holomorphic derivative wrt t for z fixed in X. As shown in [13] the fact that $\mathcal{L}(u_t)$ is affine wrt t forces the vector field V_t to be holomorphic on X for each t

and it then follows that V_t is holomorphic wrt t as well (a slight variant of this argument is recalled in section 3.6). Furthermore, as shown in [13]

$$\psi_t - S_t^*\psi_0 = C_t \tag{3.7}$$

where $\psi_t = \psi_0 + u_t$ and C_t is a constant for each t, i.e.

$$\omega_{u_t} = S_t^*\omega_0. \tag{3.8}$$

Now, by the bound 3.4 on ω_{u_t} the point-wise norm of the vector field V_t wrt the metric ω_0 is uniformly bounded in t on all of X. Hence, the equation 3.5 and a basic normal families argument applied to the family S_t yields a subsequence S_{t_j} and a holomorphic map S_1 on X such that $S_{t_j}(x) \to S_1(x)$ uniformly on X (wrt the distance defined by the metric ω_0) where S_1 is a biholomorphism according to the relation 3.8. Finally, letting $t_j \to 1$ in the relation 3.7 and using that u_t is continuous on $[0,1] \times X$ finishes the proof of the uniqueness statement in the theorem.

Remark 13. It was not explicetly pointed out in [13] that S_t lifts to L, but this fact follows from lemma 12 in [27].

3.2. **Proof of Corollary 2.** First observe that we may assume that $H^0(X, L+K_X)$ has a non-zero element (otherwise the corrollary is trivally true). But since (X, L) is homogenuous it then follows immediately that $L + K_X$ is globally generated. Hence, the conditions in Theorem 1 are satisfied.

Assume now that ω_0 is invariant under the holomorphic and transitive action of K on X. Then it follows that 0 is a critical point. Indeed, the volume form $\omega_0^n/n!$ is invariant under the action of K on X and so is the Bergman measure $\beta(0)$ (since it is defined in terms of the $K-$invariant weight ψ_0). Since the action of K is transitive and both measures are normalized it follows that the function $(\omega_0^n/n!)/\beta(0)$ on X is constant and hence equal to one. In other words, 0 is a critial point and by Theorem 1 the inequality in the statement of Corollary 2 then holds. Finally, the last statement of the corollary is a direct consequence of the uniqueness part of Theorem 1.

3.3. **Proof of Corollary 3.** Let us first prove the first statement of the corollary. Since $\mathcal{C}^\infty(X)$ is dense in $W^{1,2}(X)$ we may assume that u is smooth. First observe that

$$\mathcal{F}_{\omega_0}(u) \leq \mathcal{F}_{\omega_0}(P_{\omega_0}u). \tag{3.9}$$

To see this note that, since, by definition, $P_{\omega_0}u \leq u$ the fact that $\mathcal{L}_{\omega_0}$ is increasing immediately implies $\mathcal{L}_{\omega_0}(u) \geq \mathcal{L}_{\omega_0}(P_{\omega_0}u)$. Next, observe that by the cocycle property of $\mathcal{F}_{\omega_0}(u)$

$$\mathcal{E}_{\omega_0}(u) = \mathcal{E}_{\omega_0}(P_{\omega_0}u) + \int_X (u - P_{\omega_0}u)(\omega_u + \omega_{P_{\omega_0}u})/2$$

But, since, as is well-known the measure $\omega_{P_{\omega_0}u}$ is supported on the open set $\{u > P_{\omega_0}u\}$ (cf. Prop. 1.10 in [9] for a generalization) we have that

the last term above is equal to

$$\int_X (u-P_{\omega_0}u)(\omega_u-\omega_{P_{\omega_0}u})/2=\int_X (u-P_{\omega_0}u)(dd^c(u-P_{\omega_0}u)=$$

$$=-\int_X d(u-P_{\omega_0}u)\wedge d^c(u-P_{\omega_0}u)\leq 0,$$

where we have integrated by parts in the last equality, which is justified since, for example, by Theorem [6] $P_{\omega_0}u$ is in $\mathcal{C}^{1,1}(X)$ (but using that $P_{\omega_0}u$ is in $\mathcal{C}^0(X)$ is certainly enough by classical potential theory). Hence, $\mathcal{E}_{\omega_0}(u)\leq\mathcal{E}_{\omega_0}(P_{\omega_0}u)$ which finishes the proof of 3.9. Since, $\omega_{P_{\omega_0}u}\geq 0$ uniform approximation let's us apply Corollary 2 to deduce

$$\mathcal{F}_{\omega_0}(u)\leq\mathcal{F}_{\omega_0}(P_{\omega_0}u)\leq 0$$

which proves the first statement of the corollary.

Finally, the uniqueness will follow from Corollary 2 once we know that a maximizer u of $\mathcal{F}_{\omega_0}$ on $W^{1,2}(S^2)$ is smooth with $\omega_u>0$. By the previous step we may assume that $\omega_u\geq 0$. But since $W^{1,2}(S^2)$ is a linear space containing $\mathcal{C}^\infty(X)$ the Euler-Lagrange equations $\omega_{u_0}+dd^cu=\beta(u)$ hold for the maximizer u. Since $\beta(u)=e^{-u}\rho>0$ with ρ smooth, local elliptic estimates for the Laplacian then show that u is in fact smoth with $\omega_t>0$. All in all we have proved that

$$-\mathcal{L}_{k\omega_0}(u)\leq-\mathcal{E}_{k\omega_0}(u) \tag{3.10}$$

for $L=k\mathcal{O}(1)$ with conditions for equality.

3.3.1. *Explicit expression.* To make the previous inequality more explicit note that, by definition,

$$\mathcal{E}_{k\omega_0}(u):=\frac{1}{2\int k\omega_0}\int(udd^cu+u2k\omega_0)=\frac{1}{2k}\int udd^cu+\int u\omega_0)$$

Moreover, since for $X=\mathbb{P}^1$ we have $K_X=-\mathcal{O}(2)$ it follows that $L+K_X=\mathcal{O}(k-2)=:\mathcal{O}(m)$. Under this identification the scalar product on $H^0(X,L+K_X)$ may be written as

$$\langle s,t\rangle_{k\psi_0+u}=c\int s\bar{t}e^{-(u+m\psi_0)}\omega_0$$

using that ω_0 is a Kähler-Einstein metric, i.e. $\omega_0(z):=dd^c\psi_0=ce^{-2\psi_0}idz\wedge d\bar{z}$ for some numerical constant c. Since the funtional $\mathcal{L}$ is invariant under on overall scaling in definition of the scalar product $\langle\cdot,\cdot\rangle_\psi$ we may as well assume that $c=1$. Hence, since $N_m=m+1$, we have

$$\mathcal{L}_m(u):=(m+1)\mathcal{L}_{\omega_{0,k}}(u)=-\log\det(c_ic_j\int_{\mathbb{C}}\frac{z^i\bar{z}^j}{(1+z\bar{z})^m}e^{-u}\omega_0), \tag{3.11}$$

where $c_i=(\int\frac{\left|z^i\right|^2}{(1+z\bar{z})^m}\omega_0)^{-1/2}$. Hence, the inequality 3.10 may be expressed as

(3.12)

$$\log\det(c_ic_j\int_{\mathbb{C}}\frac{z^i\bar{z}^j}{(1+z\bar{z})^m}e^{-u}\omega_0)\leq-\frac{m+1}{(m+2)}\frac{1}{2}\int(udd^cu)-(m+1)u\omega_0.$$

In particular, when $m = 0$ the inequality above reads

$$\log(\int_{S^2} e^{-u}\omega_0) \leq \frac{1}{4}\int(udd^c u) + \int u\omega_0).$$

Finally, to compare with the notation of Onofri [41], note that, by definition, $dd^c u = \frac{i}{2\pi}\partial\bar{\partial}u$ and hence, integration by parts gives,

$$-\int udd^c u = \frac{1}{\pi}\frac{i}{2}\int \partial u \wedge \bar{\partial}u.$$

Moreover, in terms of a given local holomorphic coordinate $z = x + iy$, we have $\frac{i}{2}\partial u \wedge \bar{\partial}u = \frac{1}{4}\left|\nabla u\right|^2 dx \wedge dy$, where $\nabla = (\partial_x, \partial_y)$ is the gradient wrt the local Euclidian metric. By conformal invariance we hence obtain $-\int udd^c u = \frac{1}{4\pi}\int \left|\nabla u\right|^2 d\mathrm{Vol}_g$ for *any* Riemannian metric g on S^2 conformally equivalent to g_0. In particular, taking g as the usual round metric on S^2 induced by its embedding as the unit-sphere in Eucledian $\mathbb{R}^3$ finally gives

$$\log(\int_{S^2} e^{-u} d\mathrm{Vol}_g/4\pi) \leq \frac{1}{4}\int \left|\nabla u\right|^2 - u)d\mathrm{Vol}_g/4\pi),$$

using that $\omega_0 = d\mathrm{Vol}_g/4\pi$. This is precisely the inequality proved by Onofri [41].

Remark 14. The inequality in Corollary 3 is not only sharp in the sense that it is saturated for *some* function (for example $u = 0$), but also in the sense that if there exist constants A, B with $B \geq 0$ such that

$$-\mathcal{L}_m(u) \leq -A\int_{S^2} u\omega_0 + B\int du \wedge d^c u, \tag{3.13}$$

for all smooth u, then $A = m + 1$ and $B \geq \frac{m+1}{(m+2)}\frac{1}{2}$. Indeed, by the conditions for equality in Corollary 3 we may find a function u, which is not identically constant, saturating the inequality in Corollary 3. After adding a suitable constant to u we may assume that $\int_{S^2} u\omega_0 = 0$ and hence that

$$-\mathcal{L}_m(u) = \frac{m+1}{(m+2)}\frac{1}{2}\int_{S^2} du \wedge d^c u.$$

Now using 3.13 it follows that

$$\frac{m+1}{(m+2)}\frac{1}{2}\int_{S^2} du \wedge d^c u \leq B\int_{S^2} du \wedge d^c u,$$

i.e. that $B \geq \frac{m+1}{(m+2)}\frac{1}{2}$. Next, taking u as a constant c in 3.13 gives

$$-c(m+1) \leq -cA$$

But since c was arbitrary it follows that $A = m + 1$. In fact, a variant of the previous argument shows that Corollary 2 is sharp in a similar sense (by replacing $\int du \wedge d^c u/2$ with Aubin's $J-$functional 2.9). The details are omitted.

3.4. **Proof of Corollary 4.** We keep the notation from the previous section. For simplicity we will write $\Delta_u := \Delta^{(m)}_{\bar{\partial}_u}$. Following [32] we will first express $\det \Delta_u$ in terms of $-\mathcal{L}_m(u)$. Since the dimension $h_1(\mathcal{O}(m))$ of the first Dolbeault cohomology group $H^1(\mathbb{P}, \mathcal{O}(m))$ vanishes, the anomaly formula of Bismut-Gillet-Soulé [14] for the Quillen metric on the determinant line $\bigwedge^{N_m} H^0(\mathbb{P}, \mathcal{O}(m))$ reads as follows in our notation:

$$\log(\frac{\det \Delta_u}{\det \Delta_0}) = \int \mathrm{Td}(X, \omega_0) \wedge \tilde{ch}(e^{-u} h_0^{\otimes m}, h_0^{\otimes m}) - \mathcal{L}_m(u),$$

where $\mathrm{Td}(X, \omega_0) = (1 + A\omega_0)$ is the *Todd class* of TX represented by the constant curvature metric ω_0 expressed in terms of certain numerical constant A, and $\tilde{ch}(e^{-u} h_0^{\otimes m}, h_0^{\otimes m}) = u + (u\omega_u + \omega_0)/2$ is the *Bott-Chern class* of the two metrics $e^{-u} h_0^{\otimes m}$ and $h_0^{\otimes m}$ on $\mathcal{O}(m)$ associated to the *Chern character* of $\mathcal{O}(m)$. In fact, $A = 1$, but the actual value will turn out to be immaterial. Expanding gives

$$\log(\frac{\det \Delta_u}{\det \Delta_0}) = \int u dd^c u/2 + B \int u\omega_0 - \mathcal{L}_m(u)$$

for some constant B. Since the left hand side is invariant under translations of u by constants it follows that $B = N$. The previous formula is precisely the one appearing in Prop 1 in [32]), since $h_1(\mathcal{O}(m)) = 0$). Applying the inequality 3.12 hence gives

$$\log(\frac{\det \Delta_u}{\det \Delta_0}) \leq -\frac{1}{2}(1 - \frac{m+1}{(m+2)}) \int du \wedge d^c u = -\frac{1}{2}(\frac{1}{(m+2)}) \int du \wedge d^c u \leq 0$$

In particular, the lhs vanishes precisely when the gradient of u does, i.e. when u is a constant. This hence finishes the proof of Corollary 4.

Remark 15. In the general anomaly formula in [14] the metric ω_0 is allowed to vary as well. In particular, when $L = \mathcal{O}(0)$ is the trivial holomorphic line bundle over S^2, the metric $h = 1$ is kept constant, but the conformal metric $g_u = e^{-u} g_0$ on TS^2 varies with u, the anomaly formula in [14] is equivalent to Polyakov's formula and then $\log(\frac{\det \Delta_{g_u}}{\det \Delta_{g_0}})$ coincides with the functional $\mathcal{F}_0$ (up to a a multiplicative constant) [19].

3.5. **Arithmetic applications.** In this section we will briefly consider possible applications of Theorem 1 to Arithmetic (Arakelov) geometry in the form of *effective Riemann-Roch type inequalities.* In the general setting X will be the complex points of an *arithmetic variety* $X_{\mathbb{Z}}$ i.e. of a regular scheme, projective and flat over $\mathbb{Z}$ [48].

Consider for simplicity the case when $X = \mathbb{P}^1$ and denote as before by z the holomorphic variable in an affine piece of $\mathbb{P}^1$. Let $h^0_{L^2}(\mathcal{O}(m), u)_{\mathbb{Z}}$ denote the logarithm of the number of all polynomials p_m in z of degree at most m such that p_m has coefficients in $\mathbb{Z} + i\mathbb{Z}$ and such that $\|p_m\|^2_{u+m\psi_0} := \int_{\mathbb{C}} |p_m(z)| \, e^{-(u+m\psi_0)} \omega_0 \leq 1$. The invariant $h^0_{L^2}(\mathcal{O}(m), u)$ is a non-standard variant of basic invariants studied in Arakelov geometry, where one usually only considers sections defined over $\mathbb{Z}$ (i.e. defined by

counting those polynomials p_m as above which are invariant under complex conjugation) and *sup-norms* instead of L^2-norms (see sec. VIII, 2 in [48]). The arguments below can be adapted to such invariants, but there will be an extra non-explicit term coming from the distortion between the sup-norms and the L^2-norms determined by u.

Fixing the base of monomials (z^j) in $H^0(\mathbb{P}^1, \mathcal{O}(m))$ we may identify $H^0(\mathbb{P}^1, \mathcal{O}(m))$ with $\mathbb{C}^{N_m} = \mathbb{R}^{2N_m}$, where $N_m = m+1$. Then $h^0_{L^2}(\mathcal{O}(m), u)$ is simply the logarithm of the number of points in the standard lattice in $\mathbb{R}^{2N_m}$ contained in a convex body determined by u. Given Corollary 3, Minkowski's classical theorem is used to give an effective lower bound on $h^0_{L^2}(\mathcal{O}(m), u)$:

$$h^0_{L^2}(\mathcal{O}(m), u) \geq (m+1)\mathcal{E}_{(m+2)\omega_0}(u) + C_m,$$

where C_m is a certain explicit constant only depending on m (see below). Since the argument is standard in Arakelov geometry (compare p.164 in [48]) we will only briefly indicate it. First, by basic linear algebra, we have that $\mathcal{L}_m(u) = \log(\mathrm{Vol}\mathcal{B}(u + m\psi_0)/\mathrm{Vol}\mathcal{B}(m\psi_0))$, in terms of the volume of the unit-balls of the L^2-norms induced by the weights $u+m\psi_0$ and $m\psi_0$, respectevely, wrt Lesbegue measure in $\mathbb{R}^{2N_m}$. Denote by V_m the volume of the unit-ball in $\mathbb{R}^{2N_m}$. Then, using simple cocycle properties of $\mathcal{L}_m(u)$ (defined in formula 3.11) we get

$$\mathcal{L}_m(u) = \log(\mathrm{Vol}\mathcal{B}(u + m\psi_0) - \log V_m + Z_m,$$

where $Z_m = \log\det_{0\leq i,j\leq m}(\int_{\mathbb{C}} \frac{z^i\bar{z}^j}{(1+z\bar{z})^m}\omega_0)$. Moreover, by Minkowski's theorem (see [48])

$$h^0_{L^2}(\mathcal{O}(m), u) \geq \log(\mathrm{Vol}\mathcal{B}(u + m\psi_0) - (\log 2)(2N_m).$$

All in all this means, using Corollary 3, that

$$h^0_{L^2}(\mathcal{O}(m), u) \geq (m+1)\mathcal{E}_{(m+2)\omega_0}(u) + \log V_m - Z_m - (\log 2)(2N_m).$$

It can be checked that, when $u + m\psi_0 = m\psi$, where $dd^c\psi(z) \geq 0$ the inequality above is an asymptotic equality (this is a special case of formula 5.5). Moreover, the rhs above is equal to $m^2\mathcal{E}_{\omega_+}(\psi - \log^+|z|^2) + o(m^2)$, where $\omega_+(z) = dd^c\log^+|z|^2$. This means that the lower bound above is consistent as it must with the asymptotic arithmetic Riemann-Roch formula in [31] (see also Theorem 2' on p. 163 in [48]). In fact, the leading coefficent can be shown to coincide in this case with a (normalized) arithmetic top-intersection number, since $\mathcal{E}_{\omega_+}(\psi - \log^+|z|^2)$ is precisely the classical weighted logarithmic density of $(\mathbb{C}, \psi)$ (see [9] and references therein). The details are omitted.

3.6. **Alternative proof of uniqueness.** In this section we will show how to prove the "uniqueness" in Theorem 1 only using the regularity of the geodesics furnished by Theorem 8 and the theory of fully non-linear elliptic operators in n complex dimensions (applied to the Monge-Ampère operator on X as in [15]). In particular, this latter theory amounts to the basic *linear* elliptic estimates for the Laplacian when $n = 1$.

Recall that $W^{r,p}(X)$ denotes the Sobolev space of all distributions f on X such that f and the local derivatives of total order r are in $W^{0,p}(X) := L^p(X)$ (equvalently, all local derivatives of total order $\leq r$ are in $L^p(X)$). If f is function on $M = [0,1] \times X$ we will write $f_t \in W^{r,p}(X)$ *uniformly wrt* t if the corresponding Sobolev norms on (X, ω_0) of f_t are uniformly bounded in t. We will also use the following basic facts repeatedly:

- If f is a function on M such that $f_t \in W^{r,p}(X)$ *uniformly wrt* t, then the distribution f is in $W^{r,p}(\dot{M})$ and the corresponding Sobolev norms on $\dot{M}$ are bounded.
- Partial derivatives of distributions commute
- If $f, g \in W^{1,p}(X)$ for any $p > 1$. Then $fg \in W^{1,p}(X)$ for any $p > 1$ and Leibniz product rule holds for the distributional derivatives.

Note that as in section 3.1 it will be enough to prove that the geodesic u_t is smooth wrt (t, x) in the *interiour* of M. However, the arguments below will even give uniform estimates on the local Sobolev norms up to the boundary of M.

Assume now that the boundy data u_0 and u_1, defining the geodesic u_t are in $\mathcal{C}^{1,1}(X)$. Since u_t is convex in t the right derivative (or tangent vector) $v_t(x) := \frac{d}{dt_+} u_t$ exists for all (t, x).

Lemma 16. The right tangent vector v_t of u_t at t is uniformly bounded on M.

Proof. First observe that by the convexity in t

$$u_t - u_0 \leq t(u_1 - u_0) \leq C_1 t,$$

using that u_0 and u_1 are continuos and hence uniformly bouned on X in the last step. Hence, $v_t \leq C$. To get a lower bound first observe that there is a "psh extension" $\tilde{u}_t$ which is uniformly Lipshitz. Indeed, just take $\tilde{u}_t := (1-t)u_0 + tu_1 + Ae^t$ for $A >> 1$. Using that $0 \leq dd^c u_0, dd^c u_1 \leq C\omega_0$ it is straight-forward to check that $dd^c\tilde{U} + \pi^*\omega_0 \geq 0$ on M for A sufficently large. Since U is defined by the upper envelope 2.2 it follows that $\tilde{u}_t \leq u_t$ and hence

$$u_t - u_0 \geq \tilde{u}_t - \tilde{u}_0 \geq C_2 t.$$

giving $v_0 \geq C_2$. Finally, by convexity we get $C_2 \leq v_0 \leq v_t \leq C_1$ which proves the lemma. $\square$

Proposition 17. *Let u_0 be a critical point of $\mathcal{F}_{\omega_0}$ on $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^{1,1}(X)$, u_1 an arbitrary element in $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^{1,1}(X)$ and u_t the geodesic connecting u_0 and u_1. If $\mathcal{L}_{\omega_0}(u_t)$ is affine, then there is an automorphism S_1 of (X, L), homotopic to the identity, such that $u_1 - u_0 = S_1^*\psi_0 - \psi_0$.*

Proof. Step 1: $u_t \in \mathcal{C}^\infty(X)$. First note that by Theorem 8 $u_t \in \mathcal{C}^{1,1}_{\mathbb{C}}(X)$. Moreover, as shown in the beginning of section 3.1 it follows under the assumptions above that, for any t, the function u_t satisfies the Euler-Lagrange equations 1.6 on X. Hence, just as in section 3.1 Blocki's complex version of the regularity result of Trudinger, now applied to local

patches of $\{t\} \times X$ immediately gives that $u_t \in \mathcal{C}^\infty(X)$ (when $n = 1$ this follows from basic linear elliptic theory).

Step 2: $\Delta_X v_t \in L^\infty(X)$ *uniformly wrt* t. Differentiating the Euler-Lagrange equation wrt t from the right gives

$$ndd^c v_t \wedge (\omega_t)^{n-1} = \frac{d\beta_{u_t}(x)}{dt}_+ =: R[v_t], \tag{3.14}$$

in the sense of currents. Of course, this would follow immediately from the chain rule if u_t were smooth in (t, x). In the present case it is proved in lemma 18 below. Moreover, lemma 24 in the appendix implies the bound

$$\|R[v_t]/(\omega_0)^n\|_{L^\infty(X)} \leq C \|v_t\|_{L^\infty(X)} \tag{3.15}$$

To see this, just note that

$$R[v] \leq 2 \|v\|_{L^\infty(X)} \int_X |K(x, y)|^2 e^{-(\psi(x)+\psi(y))} = 2 \|v\|_{L^\infty(X)} \beta_u,$$

using the well-known "reproducing property" of the Bergman kernel (formula 6.3 in the appendix). By formula 6.5 in the appendix this proves the inequality 3.15.

Now, since $\omega_t > \delta\omega_0$, formula 3.14 gives that the distribution $\Delta_{\omega_t} v_t$, where Δ_{ω_t} is the Laplacian on X wrt the metric $\omega_t := \omega_{u_t}$, is in $L^\infty(X)$ uniformly wrt t and

$$\|\Delta_{\omega_t} v_t\|_{L^\infty(X)} \leq C \|v_t\|_{L^\infty(X)} \leq C',$$

by lemma 16.

Step 3: $\Delta_M u \in W^{1,p}(M)$ *for any* $p \geq 1$. First observe that by step 1

$$\partial_z(\partial_{z_i} \partial_{\bar{z}_j} u) \in L^\infty(X), \tag{3.16}$$

uniformly wrt t. Also note that

$$\partial_t(\partial_{z_i} \partial_{\bar{z}_j} u) \in L^\infty(X), \tag{3.17}$$

uniformly wrt t. Indeed, $\partial_t(\partial_{z_i} \partial_{\bar{z}_j} u) = \partial_{z_i}(\partial_{\bar{z}_j} \partial_t u) = (\partial_{z_i} \partial_{\bar{z}_j}) v_t \in L^p(X)$, uniformly wrt t, for any $p > 1$, by step 2 and local elliptic estimates for Δ_X. Next, we will use that the following identity proved in lemma 19 below:

$$\partial_t \partial_{\bar{t}} u = |V_t|^2_{\omega_t} = |\partial_{\bar{z}} v_t|^2_{\omega_t}, \tag{3.18}$$

where $|V_t|^2_{\omega_t}$ denotes the point-wise norm of V_t wrt the metric ω_t (where we have used that $\omega_t > 0$). First we have

$$\partial_z(\partial_t \partial_{\bar{t}} u) = \partial_z |\partial_{\bar{z}} v_t|^2_{\omega_t} \in L^p(X), \tag{3.19}$$

uniformly wrt t, for any $p > 1$ using Step 1 and Step 2 combined with local elliptic estimates on X for Δ_X. Next,

$$\partial_t(\partial_t \partial_{\bar{t}} u) \in L^p(X), \tag{3.20}$$

uniformly wrt t. Indeed, $\partial_t(\partial_t\partial_{\bar{t}}u) = \partial_t \left|\partial_{\bar{z}}\partial_t u\right|^2_{\omega_t}$ and since locally $\partial_t\omega_t = \partial_t(\partial_z\partial_{\bar{z}}u)$ 3.20 follows from 3.17 and 3.19 combined with Leibniz product rule. All in all this proves Step 3.

Now by Step 3 and elliptic estimates for the Laplacian we have $u \in W^{3,p}(M)$. In particular, u is locally in $\mathcal{C}^2(M)$. As a consequence the proof of Theorem 2.6 in [13] immediately gives that V_t is a holomorphic vector field on X for any t. Finally, we will recall a slight variant of the argument in [13] which shows that $\partial_{\bar{t}}V_t = 0$ for V_t seen as a distribution on the interiour of M. To simplify the notation we assume that $n = 1$, but modulo the change to matrix notation the case $n > 1$ is the same. First we write 3.6 in the form

$$\omega V = \partial_{\bar{z}}\partial_t u, \tag{3.21}$$

where we have identified V and ω with elements in $L^p(M)$ for $p >> 1$. By Leibniz rule

$$\partial_{\bar{t}}(V\omega) = (\partial_{\bar{t}}V)\omega + V(\partial_{\bar{t}}\omega)$$

Next, observe that

$$\partial_{\bar{t}}\omega = \partial_{\bar{t}}(\partial_z\partial_{\bar{z}}u) = \partial_{\bar{z}}(\partial_{\bar{t}}\partial_z u) = \partial_{\bar{z}}(\omega\bar{V}),$$

using 3.21 in the last step. Hence, since, as shown above, $\partial_{\bar{z}}V = 0$, the two previous equations together give

$$\partial_{\bar{t}}(V\omega) = (\partial_{\bar{t}}V)\omega + \partial_{\bar{z}}(V\omega\bar{V}) = \partial_{\bar{z}}(\partial_{\bar{t}}\partial_t u),$$

also using 3.21 in the last step and commuting $\partial_{\bar{z}}$ and $\partial_{\bar{t}}$. Since, $V\omega\bar{V} = |V|^2_\omega$ it follows by 3.18 that $(\partial_{\bar{t}}V)\omega = 0$. But since, $\omega > 0$ and $(\partial_{\bar{t}}V)$ is in $L^p(M)$ for all $p > 1$ this forces $(\partial_{\bar{t}}V) = 0$ a.e. on M. In particular, $(\partial_{\bar{t}}V) = 0$ as a distribution on M. Hence, it follows that the distribution V_t is in the null-space of the $\bar{\partial}-$operator on M. By local elliptic theory it follows that V_t is smooth and hence holomorphic in the interiour of M. Finally, the automorphism S_1 is obtained precisely as in the end of section 3.1. ☐

Lemma 18. Under the assumptions in the previous proposition the following holds:

$$\frac{d}{dt_+}\int_X(\omega_t)^n f = \int_X nv_t\wedge(\omega_t)^{n-1}\wedge dd^c f,$$

where f is a given smooth function on X.

Proof. To simplify the notation we assume that $n = 2$ and $t = 0$, but the general argument is completely similar (compare [10]). Expanding and using that (by convexity) $(u_t - u_0)/t$ decreases point-wise to u_0, shows that it is equivalent to prove

$$\int(u_t - u_0)(dd^c(u_t - \phi_0))\wedge dd^c f = o(t).$$

By partial integration the l.h.s is equal to

$$-\int d(u_t - u_0)\wedge d^c(u_t - u_0)\wedge dd^c f$$

Taking absolute values and using that $d(u_t - u_0) \wedge d^c(u_t - u_0) \geq 0$ pointwise shows in turn that it is enough to prove the following

$$\text{(3.22)} \qquad \text{Claim: } \int d(u_t - u_0) \wedge d^c(u_t - u_0) \wedge \omega_0 = o(t)$$

To this end first observe that

$$\text{(3.23)} \qquad \frac{d}{dt}_{t=0+} \mathcal{E}(u_t) = \int_X v_0 (\omega_0)^n / (n! V)$$

To see this, note that since we have already shown that $\mathcal{E}(u_t)$, $\mathcal{F}(u_t)(= \mathcal{E}(u_t) - \mathcal{L}(u_t))$ *and* $\mathcal{L}(u_t)$ are all affine (and even constant)

$$\text{(3.24)} \qquad 0 = \frac{d}{dt}_{t=0+} \mathcal{E}(u_t) = \frac{d}{dt}_{t=0+} \mathcal{L}(u_t) = \int_X v_0 \beta_{u_t},$$

using formula 2.3 in the last step (and dominated convergence for the sequence $(u_t - u_0)/t$ converging to v_0). Since, u_t satisfies the Euler-Lagrange equations 1.6 this proves formula 3.23. □

Next, we will use the following well-known general identity (see [10] or page 58-59 in [51]):

$$\mathcal{E}(u_t) - \mathcal{E}(u_0) - \int (u_t - u_0)(\omega_0)^n / n! = -J_{\omega_0}(\phi_t),$$

in terms of the non-negative functional

$$J_{\omega_0}(u_t) = c_1 \int d(u_t - u_0) \wedge d^c(u_t - u_0) \wedge \omega_0 + c_2 \int d(u_t - u_0) \wedge d^c(u_t - u_0) \wedge \omega_t$$

where $c_i > 0$ (compare formula 2.9). But by 3.24 and the identity above

$$\frac{d}{dt}_{t=0+} J_{\omega_0}(\phi_t) = 0,$$

which by positivity implies the claim in formula 3.22 and hence finishes the proof of the lemma.

In the previous proof we also used the following

Lemma 19. Under the assumptions in the previous proposition the following holds: $\partial_t \partial_{\bar{t}} u \in L^\infty(X)$ uniformly in t and

$$\partial_t \partial_{\bar{t}} u = \left| \bar{\partial}_X \partial_t u \right|^2_{\omega_{u_t}}.$$

Proof. By assumption the Monge-Ampère measure $(dd^c U + \pi_X^* \omega_0)^{n+1}$ vanishes on M. Moreover, by Step 1 in the proposition above $\Delta_X u_t \in C^\infty(X)$ for any t with bounds on the Sobolev norms which are uniform wrt t. Combining this latter fact with lemma 16 gives that U is Lipschitz on M. Finally, as shown in Step 2 in the proof of proposition above $\Delta_X \partial_t u_t \in L^\infty(X)$ uniformly wrt t. We will next show that these properties are enough to prove the lemma. As the statement is local we may as well consider the restriction of $u := U$ to an open set biholomorphic to a domain in $\mathbb{C}^{n+1} = \mathbb{C}_t \times \mathbb{C}^n_z$. Denote by u^ϵ the local smooth function

obtain as the convolution of u with a fixed local compactly supported smooth family of approximations of the identity. Expanding gives

$$(dd^cU + \pi_X^*\omega_0)^{n+1} = (\partial_t\partial_{\bar{t}}u^\epsilon - |\partial_{\bar{z}}\partial_t u^\epsilon|^2_{\omega_{u^\epsilon}})(\omega_{u^\epsilon})^n \wedge dt \wedge d\bar{t}. \tag{3.25}$$

Now since, by assumption, $|\partial_{\bar{z}}\partial_t u^\epsilon|^2_{\omega_{u^\epsilon}} \leq C$ the second term tends to $|\partial_{\bar{z}}\partial_t u|^2_{\omega_u})(\omega_u)^n \wedge dt \wedge d\bar{t}$ weakly when $\epsilon \to 0$. Moreover, by assumption $u^\epsilon \to u$ uniformly locally and since the Monge-Ampère operator is continous, as a measure, under uniform limits of psh functions [23] it will now be enough to prove that

$$(\partial_t\partial_{\bar{t}}u^\epsilon)(\omega_{u^\epsilon})^n \wedge dt \wedge d\bar{t} \to (\partial_t\partial_{\bar{t}}u)(\omega_u)^n \wedge dt \wedge d\bar{t} \tag{3.26}$$

weakly, where the right hand sice is well-defined since $\partial_t\partial_{\bar{t}}u_t$ defines a positive measure on $\mathbb{C}^{n+1}$ and $(\omega_{u_t})^n/\omega_0^n$ is continous on $\mathbb{C}^{n+1}$. To this end fix a test funtion f i.e. a smooth and compactly supported function on $\mathbb{C}^{n+1}$. Then, with $\int$ denoting the integral over $\mathbb{C}^{n+1}$,

$$\int f(\omega_{u^\epsilon})^n(\partial_t\partial_{\bar{t}}u^\epsilon) \wedge dt \wedge d\bar{t} =: \int g_\epsilon(\partial_t\partial_{\bar{t}}u^\epsilon) = -\int(\partial_t g_\epsilon)(\partial_{\bar{t}}u^\epsilon)$$

By assumption $(\partial_t g_\epsilon)$ and $(\partial_{\bar{t}}u^\epsilon)$ tend to $(\partial_{\bar{t}}u)$ and $(\partial_{\bar{t}}u)$, respectively in $L^p(X)$ for any $p > 1$, uniformly wrt t (more precisily by the assumption on $\Delta_X u_t$ and the fact that u is Lipschitz). Hence, by Hölders's inequality

$$\int g_\epsilon(\partial_t\partial_{\bar{t}}u^\epsilon) \to -\int(\partial_t g)(\partial_{\bar{t}}u).$$

Finally, since $(\partial_t g) \in L^\infty(X)$ uniformly wrt t (by the assumption on $\Delta_X u_t$) and since $\partial_t\partial_{\bar{t}}u$ defines a positive measure, Leibniz rule combined with the dominated convergence theorem gives (by a simple argument using a regularization of g)

$$-\int(\partial_t g)(\partial_{\bar{t}}u) = \int g(\partial_t\partial_{\bar{t}}u)$$

This proves 3.26 and hence finishes the proof of the lemma. □

4. Application to $SU(2)-$invariant determinantal point processes

A random point process with N particles on a space X wrt to a *background measure* μ on X, may be definied as an ensemble of the form (X^N, γ_N), where

$$\gamma_N = \rho_N(x_1, ..., x_N)d\mu^{\otimes N}$$

and where the density $\rho_N(x_1, ..., x_N)$ of the probability measure γ_N is assumed invariant under the action of the symmetric group S_N, i.e. under permutations of the x_i :s. The N-fold product X^N is called the *$N-$particle configuration space.* The random point process (X^N, γ_N) determines the random measure

$$(x_1, ..., x_N) \mapsto \sum_{i=1}^N \delta_{x_i}, \tag{4.1}$$

(i.e. a measure valued random variable) often called the *empirical measure.* Given a, say continuous, function u on X one defines the corresponding *linear statistic* as the random variable obtained by contraction with the empirical measure:

$$(x_1, ..., x_N) \mapsto \sum_{i=1}^{N} u(x_i) \tag{4.2}$$

Using standard probability notation we will write $\mathbb{E}_N(Y) := \int Y d\gamma_N$ for the *expectation* of a random variable Y on X^N and its *fluctuation* $\tilde{Y}$ is then the centered random variable $\tilde{Y} - \mathbb{E}_N(Y)$. We also write $\mathrm{Prob}_N(A) := \int_A \gamma_N$.

A special class of random point processes are given by the *determinantal* ones [35], which exhibit repulsion. . These have been mainly studied when the background measure μ supported on $\mathbb{C}$; notably in the context of random matrix theory (cf. [24]). A general complex geometric framework for determinatal random point processes was introduced in [7]. Given a line bundle $L \to X$ over a compact complex manifold X, a background measure μ and a weight ψ_0 of a metric on L, the corresponding point process is obtained by setting

$$\rho_N(x_1, ..., x_N) := \left| \det_{1 \leq i,j \leq N} (s_i(x_i))_{i,j} \right|^2 e^{-\psi_0(x_1)} ... e^{-\psi_0(x_N)} / Z$$

in terms of any base $S = (s_i)$ for the Hilbert space $H^0(X, L)$ equipped with the scalar product induced by (ψ_0, μ). The number Z (called the *partition function*) is the normalizing constant ensuring that γ_N is a probability measure on X^N. Even if Z does depend on the base S the density ρ_N does not. In the "adjoint" setting considered in the present paper where L is replaced by $L + K_X$, there is no need to specify the background measure μ_0 (equivalenly, μ_0 is taken as any smooth volume form on X which induces an Hermitian metric on K_X in such a way that the density ρ_N is independent of μ_0).

The bridge between the point above processes and the subject of the present paper is furnished by a formula which is a simple variant of a well-known formula of Heine and Szegö in the theory of orthogonal polynomials:

$$\mathbb{E}_N(e^{-(u(x_1)+...+u(x_n))}) = \det_{1 \leq i,j \leq N} \langle s_i, s_j \rangle_{(\psi_0 + u, \mu)} \ , \tag{4.3}$$

i.e. $-\log \mathbb{E}_N(e^{-(u(x_1)+...+u(x_n))}) = \mathcal{L}_{\omega_0}(u)$ in the notation of the previous sections [7]. In fact, the formula above is a simple consequence of the following identity

$$\int_{X^N} \left| \det_{1 \leq i,j \leq N} (s_i(x_i))_{i,j} \right|^2 e^{-\psi_0(x_1)} ... e^{-\psi_0(x_N)} \mu = N!,$$

for (s_i) an orthonormal wrt the Hermitian products induced by (ψ_0, μ). Note that in the probability litterature $\mathbb{E}(e^{tY})$ is called the *moment generating function* of a given random variable Y.

4.1. **The case of the two-sphere.** Let now X be the two-sphere S^2 embedded as the unit-sphere in Euclidian $\mathbb{R}^3$ and set

$$\rho_N(x_1, ..., x_N) := \Pi_{1 \leq i < j \leq N} \|x_i - x_j\|^2 / Z_N \tag{4.4}$$

written in terms of the ambient Euclidian norm in $\mathbb{R}^3$, where Z_N is the normalizing constant ensuring that γ_N is a probability measure on X^N [in fact $1/Z_N = N^N \binom{N-1}{0} ... \binom{N-1}{N-1}/N!$] The background measure is taken as the induced volume (or rather area) form ω_0 on S^2 normalized to give unit volume to S^2. Note that formula 4.6below shows that $g(x, y) := \log \|x - y\|^2$ is the *Green function* for (S^2, ω_0) (compare section 2.1 in [8] and section 4 in [52])

This random point process has two crucial properties: $i)$ it is invariant under the isometry group of S^2 and $ii)$ it is *determinantal.*

In the physics litterature the ensemble above appears as the Gibbs ensemble of a *Coulomb gas* of unit-charge particles (i.e one component plasma) confined to the sphere [17]. An interesting *random matrix* realiziation of this process was found very recently in [38] (compare remark 22 below).

In this probabilistic frame work Corollary 3 may now be formulated as the following "multi-particle Moser-Trudinger inequality" on S^2 (which is sharper then the one conjectured in section 5 in [30]).

Theorem 20. *The following upper bound on the moment generating function of the fluctuation of a linear statistic in the point process 4.4 with $N-$particles on S^2 holds*

$$\log \mathbb{E}_N(e^{t(\widetilde{u(x_1)} + \widetilde{u(x_2)} + ... + \widetilde{u(x_N)})}) \leq \frac{N}{N+1} \frac{t^2}{2} \|du\|^2 \tag{4.5}$$

for any $t \in \mathbb{R}$ with equality iff $\omega_0 - tdd^c u$ is the pull-back of ω_0 under a conformal transformation of S^2.

Proof. First observe that, in terms of the standard complex coordinate z on S^2 with the north pole removed we have the basic identity

$$\|x_1 - x_2\|^2 = |z_i - z_j|^2 e^{-\psi_0(z_1)} e^{-\psi_0(z_2)}, \tag{4.6}$$

(this is obvious for z_i and z_j on the unit-circle in $\mathbb{C}$ and hence it holds everywhere, since the action of the group $SU(2)$ by Möbius transformations acts transitively and preserves both sides above). Substituting the previous formula in the definition of ρ_N above shows, using the standard product formula for the Vandermonde determinant $\Delta(z_1, ..., z_N)$, that

$$\rho_N(x_1, ..., x_N) := |\Delta(z_1, ..., z_N)|^2 e^{-\psi_0(z_1)} ... e^{-\psi_0(z_N)},$$

where $\Delta(z_1, ..., z_{N_k}) = \det(s_i(z_j))$, with $s_i(z)$ equal to the monomial z^j. By the general formula 4.3 it follows that

$$-\log \mathbb{E}_N(e^{-(u(x_1)+...+u(x_N))}) = \mathcal{L}_{N-1}(u)$$

where $\mathcal{L}_m(u)$ is the functional 3.11 . A simple scaling hence gives

$$\log \mathbb{E}_N(e^{-t(u(\tilde{x_i})+...+u(\tilde{x_N})}) = -\mathcal{L}_{N-1}(u) - \mathbb{E}_N(\sum_{i=1}^N u(x_i)).$$

Moreover, by general properties of determinantal point processes there exists a function ρ_1 (called the one-point correlation function [35]) on X such that

$$\mathbb{E}_N(\sum_{i=1}^N u(x_i)) = \int u\rho_1\omega_0,$$

(in the present setting $\rho_1\omega_0$ may be identified with the Bergman measure β_0 corresponding to $u=0$; compare formula 6.4). Since ρ_N and hence ρ_1 is invariant under isometries of S^2 it follows that ρ_1 is identically constant. Setting $u=1$ above forces in turn this constant to be equal to N. All in all this means that

$$\log \mathbb{E}_N(e^{-t(u(\tilde{x_1})+...+u(\tilde{x_N})}) = -\mathcal{L}_{N-1}(u) - N\int u\omega_0$$

Hence, applying Corollary 3 finishes the proof of the theorem (by replacing u by $-tu$). $\square$

In the formula above we used the notation $\|du\|^2$ for the squared L^2-norm on S^2 of the gradient of u, written in conformally invariant notation as $\|du\|^2 := \int_{S^2} du\wedge d^c u$ as in previous sections. Since the moment generating function of a random variable controls the tail of its distribution we obtain the following effective large deviation bound:

Corollary 21. *In the setting of the previous theorem the following large deviation bound holds: for any given positive number λ:*

$$Prob_N\{\frac{1}{N}(u(x_1)+...+u(x_N))>\lambda\} \leq e^{-\frac{N^2\lambda^2}{2\|du\|^2}\frac{N+1}{N}}$$

if the linear statistic 4.2 is centered, i.e. if its expected value vanishes.

Proof. The proof of this consequence of the previous theorem is a standard application of Markov's inequality: for any given $t>0$ we have

$$\text{Prob}\{Y>1\} = \text{Prob}\{e^{tY}>e^t\} \leq e^{-t}\mathbb{E}(e^{tY}),$$

where in our case $Y = \frac{1}{N\lambda}(u(x_1)+...+u(x_N))$. By the previous theorem the rhs above is bounded by e^{-t+ct^2} for $c = \frac{N}{N+1}\frac{1}{2}\left\|d(\frac{1}{N\lambda}u)\right\|^2$. Setting $t=1/2c$ (i.e. optimizing over t) finally proves the corollary. $\square$

Note that effective bounds as above are usually called *Chernoff bounds* in the classical probabilistic setting where the role of the linear statistic is played by a random variable Y of the form $Y=\frac{1}{N}(Y_1+...+Y_N)$, where Y_i are *independent* random variables with identical *symmetric* distribution.

The bound in the previous corollary should be compared with the general *non-effective* bound

$$\text{Prob}_{N_k}\{\frac{1}{N}(u(x_1)+...+u(x_N))>\lambda\} \leq Ce^{-N^2/C}, \tag{4.7}$$

where C is a non-explicit constant, implied by the *large deviation principle* proved in [8] for determinantal point process in the general line bundle setting (compare the beginning of this section). Note also that the bound 4.7 is essentially contained in the analysis in [52], since $X = \mathbb{P}^1$ in this case.

In the large $N-$limit the inequality in the previous theorem is also closely related to a *Central Limit Theorem (CLT)* for the linear statistic 4.2. Indeed, when N tends to infinity it can be shown that the inequality 4.5 becomes an asymptotic *equality, i.e.*

$$\lim_{N\to\infty} \log \mathbb{E}_N(e^{-t(\tilde{u(x_i)}+...+\tilde{u(x_N)})}) = \frac{t^2}{2}\int_{S^2} du \wedge du^c \tag{4.8}$$

for any $t \in \mathbb{R}$. In turn, by basic probability theory, this latter fact can be shown to be equivalent to the following CLT:

$$\widetilde{u(x_1)} + \widetilde{u(x_2)} + ... + \widetilde{u(x_N)} \to \mathcal{N}(0, \frac{1}{2}\left\|du\right\|^2),$$

in distribution, when $N \to \infty$, where $\mathcal{N}(0, \frac{1}{2}\left\|du\right\|^2)$ is the centered normal variable with variance $\frac{1}{2}\left\|du\right\|^2$. See [47] for combinatorial proofs of this CLT on the sphere and [7] for general results in the line bundle setting, using Bergman kernel asymptotics. It is also interesting to compare with the case of unitary random matrices, where the the role of the asymptotics 4.8 is played by *Szegö's strong limit theorem* [24]. See also [36] for the case of *Hermitian* random matrices and [2] for *normal* random matrices.

Loosely, speaking the CLT theorem above may also be formulated as the statement that the *the potential of the fluctuations of the empirical measure 4.1 on S^2 converges in distribution to the Gaussian free field on* S^2 (see the introduction in [47] and references therein).

Remark 22. Consider the probability measure on $gl(N, \mathbb{C})$ obtained by declaring the complex entries of an $N \times N$ matrix to be i.i.d complex Gaussians. Let Φ_N be the map defined by

$$\Phi_N : (G_1, G_2) \mapsto (z_1, ..., z_N)/S_N,$$

where the z_i :s are the N zeroes in $\mathbb{C}$ (taking multiplicities into account) of $\det(G_1 - zG_2)$, i.e. the eigen values of the matrix $G_2(G_1)^{-1}$, when G_1 is invertible. A remarkable result in [38] says that the push-forward under Φ_N of the product probability measure on $gl(N, \mathbb{C}) \times gl(N, \mathbb{C})$ is precisely the random point process on S^2 with N particles defined by the density 4.4 (under stereographic projection).

5. Convergence towards Mabuchi's K-energy

In this section we will briefly consider the asympotic situation when the ample line bundle L is replaced by a multiple kL for a large positive integer k. Building on [13] Berndtsson we will relate the large k asymptotics of $\mathcal{F}_{k\omega_0}$ to *Mabuchi's K-energy.* The work [13] was in turn inspired

by the seminal work of Donaldson [28] where a functional closely related to $\mathcal{F}_{k\omega_0}$ was introduced (see section 5.1 below). It should be pointed out that there will be no original results in this section. But we will give a simple proof of Theorem 23 below which only uses the C^0-regularity of the geodesic connecting two given smooth points in $\mathcal{H}_{\omega_0}$, which hopefully is of some interest. See [19] for a proof which uses the $\mathcal{C}^{1,,1}_{\mathbb{C}}-$regularity (Theorem [19]) in the case when the first Chern class of X is assumed non-positive).

Fixing $\omega_0 \in c_1(L)$ we willl take $k\omega_0$ as the reference Kähler metric in $c_1(kL)$. Throughout the section u will denote an element in $\mathcal{H}_{\omega_0}$. For simplicity we assume that that the volume $\mathrm{Vol}(\omega_0) = 1$. We will write

$$\mathcal{F}_k(u) := k\mathcal{F}_{k\omega_0} - \bar{s}\mathcal{E}_{\omega_0}$$

where $\bar{s}$ is the topological invariant of L defined as the average of the *scalar curvature* s_u of the Kähler metric ω_u for any u in $\mathcal{H}_{\omega_0}$. The scaling in the definition of $\mathcal{F}_k(u)$ is motivated by the fullowing asymptotic expansion of the differential of the functional $\mathcal{L}_{k\omega_0}$:

$$(d\mathcal{L}_{k\omega_0})_{ku} = \omega_u^n(1 + \frac{1}{k}s_u + o(1))/n! \tag{5.1}$$

where the term $o(1)$ denotes a function which tends to zero uniformly on X (for u fixed).

Using formula 2.3 the proof of the previous formula is reduced to the well-known asymptotics of the Bergman measure on $kL + E$, where E is a given line bundle on E, due to Tian-Catlin-Zelditch. The reason that s_u appears in the second term is that $E = K_X$ (see [13]). In particular, we obtain

$$(d\mathcal{F}_k)_{ku} := -(s_u - \bar{s})\omega_u^n + o(1) \tag{5.2}$$

Following Mabuchi [44, 51] the *K-energy* (also called the *Mabuchi functional*) is defined, up to an additive constant, as the primitive $\mathcal{M}$ on $\mathcal{H}_{\omega_0}$ of the exact one-form defined by the measure valued function $u \mapsto (s_u - s)\omega_u^n$ on $\mathcal{H}_{\omega_0}$. Hence, u is a critical point of $\mathcal{M}$ on $\mathcal{H}_{\omega_0}$ iff the Kähler metric ω_u has *constant scalar curvature.* We will denote by $\mathcal{M}_{\omega_0}$ the K-energy normalized so that $\mathcal{M}_{\omega_0}(0) = 0$. Integrating along line segments in $\mathcal{H}_{\omega_0}$ and using 5.2 immediately gives the asympotics

$$\mathcal{F}_k(u) = -\mathcal{M}_{\omega_0}(u) + o(1). \tag{5.3}$$

For the most general version of the following theorem see [21].

Theorem 23. *Assume that the Kähler metric ω_u has constant scalar curvature. Then u minimizes Mabuchi's K-energy $\mathcal{M}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$.*

Proof. By the cocycle property of $\mathcal{M}_{\omega_0}$ we may as well assume that $u = 0$ in the statement above. Now fix an arbitrary u in $\mathcal{H}_{\omega_0}$ and take the C^0-geodesics u_t connecting 0 and u. Given a positive integer k the fact

that $\mathcal{F}_k$ is concave along u_t (compare the proof of Theorem 1) immediately gives

$$\mathcal{F}_k(u) \leq \mathcal{F}_k(0) + \frac{d}{dt}_{t=0+}\mathcal{F}(u_t).$$

Combining formulas 5.3, 5.2 then gives

$$\mathcal{F}_k(u) \leq -\mathcal{M}_{\omega_0}(u) + \int_X (s_u - \bar{s})\omega_u^n v_0 + \int_X o(1)\omega_u^n v_0,$$

where $v_0 = \frac{du}{dt}_{t=0+}$. But by lemma 16 we have that v_0 is uniformly bounded (in fact it is enough to know that its L^1-norm is uniformly bounded, which can be proved as in [10]. Letting k tend to infinity the assumption on u hence gives

$$-\mathcal{M}_{\omega_0}(u) \leq -\mathcal{M}_{\omega_0}(0),$$

which hence finishes the proof of the theorem. □

In particular, the proof above shows that, $\mathcal{M}_{\omega_0}$ is "convex along a geodesic", in the sense that it is the point-wise limit of the *convex* functionals $\mathcal{F}_k$ along a geodesic connecting two points in $\mathcal{H}_{\omega_0}$, only using the C^0-regularity of the corresponding geodesic. Note however that the definition of $\mathcal{M}_{\omega_0}$ as given above does not even make sense unless u_t is in $\mathcal{C}^4(X)$, for t fixed and $\omega_t > 0$ (the smoothness assumption may be relaxed to $u_t \in \mathcal{C}^{1,1}_{\mathbb{C}}(X)$ using the alternative formula for $\mathcal{M}_{\omega_0}$ from [50, 20]). In the case when the geodesic u_t is assumed *smooth* and $\omega_t > 0$ the argument in the proof of the theorem above is essentially contained in [13]. In this latter case the convexity statement seems to first have appeared in [43] (see also [26]). In [28] the previous theorem was proved using the deep results in [27] and the "finite dimensional geodesics" in approximations of $\mathcal{H}_{\omega_0}$ as briefly explained in the following section.

5.1. **Comparison with Donaldson's setting and balanced metrics.** In the setting of Donaldson [28] the role of the space $H^0(X, L+K_X)$ is played by the space $H^0(X, L)$. Any given function u in $\mathcal{H}_{\omega_0}$ induces an Hermitian norm $Hilb(u)$ on $H^0(X, L)$ defined by

$$Hilb(u)[s]^2 := \int_X |s|^2 e^{-(\psi_0+u)}(\omega_u)^n/n!$$

Then the functional that we will refer to as $\mathcal{L}_D(u)$, which plays the role of $\mathcal{L}_{\omega_0}(u)$ in Donaldson's setting, is defined as in formula 1.4, but using the scalar product on $H^0(X, L)$ corresponding to $Hilb(u)$. With this definition it turns out that $\mathcal{L}_D(u)$ is *concave* along smooth geodesics (see Theorem 3.1 in [13] for a generalization of this fact). However, it does not appear to be concave along a general psh paths, which makes approximation more difficult in this setting. Moreover, Theorem 2 in [28] says that the critical points of $\mathcal{E} - \mathcal{L}_D$ are in fact *minimizers.* [1]A major technical advantage of Donaldson's setting is that the critical points

[1]Comparing with the notation in [28], $\mathcal{L}_D$, $\mathcal{E}$ and u correspond to $-\mathcal{L}$, $-I$ and $-\phi$, respectively.

(which are called *balanced* in [28]) of the functional $\mathcal{E}-\mathcal{L}_D$ acting on all of $\mathcal{C}^\infty(X)$ are automatically of the form

$$\psi=\log(\tfrac{1}{N}\textstyle\sum_i|S_i|^2) \tag{5.4}$$

fore some base (S_i) in $H^0(X,L)$. In particular, u is automatically in $\mathcal{H}_{\omega_0}$ (assuming that L is very ample). This is then used to replace the space $\mathcal{H}_{\omega_0}$ by the sequence of *finite dimensional* symmetric spaces $GL(N,\mathbb{C})/U(N)$ corresponding to the set of metrics on L of the form 5.4 (called Bergman metrics). In particular, the new geodesics, defined wrt the Riemannian structure in the symmetric space $GL(N,\mathbb{C})/U(N)$ are automatically smooth and the analysis in [28] is reduced to this finite dimensional situation.

Note also that in this setting there is a sign difference in the expansion 5.1, where s_u is replaced by $-s_u$. As a consequence, in Donaldson's case the functional corresponding to $\mathcal{F}_k$ converges to $\mathcal{M}_{\omega_0}$ (without the minus sign!), which hence becomes convex along smooth geodesics, which is consistent with the conclusion reached above, as it must.

Finally, note that combining the upper bound in Theorem 1 combined with the lower bound coming from a (slight variant) of Donaldson's scalar product on $H^0(X,L+K_X)$ (i.e. using Theorem 2 in [28]) gives

$$-C+k\mathcal{E}_{k\omega_0}(u)\leq\mathcal{L}_{k\omega_0}(ku)\leq k\mathcal{E}_{k\omega_0}(u),$$

where C is a positive constant proportional to $\|\omega_u^n/\omega_0^n\|_{L^\infty(X)}$. In particular, this yields the asymptotics

$$\mathcal{L}_{k\omega_0}(ku)=k\mathcal{E}_{\omega_0}(u)+O(1), \tag{5.5}$$

which is a well-known result. In fact, it may be directly obtained using the leading term in the asymptotics 5.1(see [9] for the generalization to non-positively curved metrics).

6. Appendix

6.1. **Bergman kernels.**

Given a function u corresponding to the weight $\psi:=\psi_0+u$ on the line bundle L we denote by $K_u(x,y)$ the *Bergman kernel* of the Hilbert space $(H^0(X,L+K_X),\langle\cdot,\cdot\rangle_{\psi_0+u})$, i.e.

$$K_u(x,y):=i^{n^2}\sum_{i=1}^N s_i(y)\wedge s_i\bar{(x)},$$

represented in terms of a given orthonormal base (s_i) in $(H^0(X,L+K_X),\langle\cdot_i,\cdot\rangle_{\psi_0+u})$. This kernel may be caracterized as the integral kernel of the correponding orthogonal projection Π_u onto $(H^0(X,L+K_X),\langle\cdot_i,\cdot\rangle_{\psi_0+u})$, i.e. for any smooth section s of $L+K_X$

$$(\Pi_u s)(x)=\int_{X_y}s(y)\wedge\bar{K}(x,y)e^{-(\psi(y)} \tag{6.1}$$

The *Toeplitz operator* $T[f]$ *with symbol* $f \in C^0(X)$, acting on $(H^0(X, L+K_X), \langle \cdot_i, \cdot \rangle_{\psi_0})$ (defined below formula 1.5) may then be expressed as

$$(6.2) \qquad (T[f])(x) = \int_{X_y} f(y)s(y) \wedge \bar{K}(x,y)e^{-\psi(y)}$$

Applying 6.1 $K_u(x, \cdot)$ gives the following "integrating out" formula

$$(6.3) \qquad N\beta_u(x) := K_u(x,x)e^{-\psi(x)} := \int_{X_y} |K(x,y)|^2 e^{-(\psi(x)+\psi(y))}$$

When studying the dependence of β_u on u it is useful to express $\beta_u(x)$ as the normalized *one-point correlation measure* of the determinantal point process induced by ψ (see section 4 and [7])
(6.4)

$$\beta_u(x) = \frac{1}{N}\mathbb{E}_\psi(\sum_{i=1}^N \delta_{x_i}) = \int_{X^{N-1}} |(\det S_0)(x, x_2, ..., x_N|^2 e^{-\psi(x)}e^{-\psi(x_2)}...e^{-\psi(x_N)}/Z_\psi$$

In particular, the map $(x,t) \mapsto (\beta_{u_t}(x)/\omega_0^n)$ is *continuous* if u_t is a continuous path and hence there is a positive constant C such that

$$(6.5) \qquad 1/C \leq (\beta_{u_t}(x)/\omega_0^n) \leq C$$

on $[0,1] \times X$, if $L + K_X$ is globally generated, i.e. if $\beta_{u_t}(x) > 0$ pointwise. Formula 6.4 also shows, by the dominated convergence theorem, that $\frac{d\beta_{u_t}(x)}{dt}_{t=0+}$ exists under the assumptions in the following lemma.

Lemma 24. *Let u_t be a family of continuous functions on X such that the right derivative $v_t := \frac{du_t}{dt}_+$ exists and is uniformly bounded on $[0,1] \times X$. Then*
(6.6)

$$R[v](x) := \frac{d\beta_{u_t}(x)}{dt}_{t=0+} = \int_{X_y} |K_u(x,y)|^2 e^{-(\psi_0(x)+\psi_0(y))}v_0(y) - \beta_{u_t}(x)v_0(x)$$

.

Proof. The proof of the formula was obtained in [12] (formula 5), at least in the smooth case. For completeness we recall the simple proof. By the discussion above we may differentiate formula 6.3 and use Leibniz product rule to get

$$\partial_t(K_t(x,x)) = 2\text{Re} \int_{X_y} \partial_t(K_t(x,y) \wedge \bar{K}(x,y)e^{-\psi_t(y)} - \int_{X_y} |K_t(x,y)|^2 (\partial_t\psi_t(y))e^{-(\psi_t(x)+\psi_t(y)}$$

Applying formula 6.1 to the holomorphic section $s(\cdot) = \partial_t K_t(x, \cdot)$ shows that the second term above equals $2\partial_t(K_t(x,x))$. Hence,

$$\partial_t(K_t(x,x)) = \int_{X_y} |K_t(x,y)|^2 (\partial_t u_t(y))e^{-(\psi_t(x)+\psi_t(y)},$$

which proves the lemma, since $N\beta_u(x) = K(x,x)e^{-\psi(x)}$. $\square$

6.2. **A "Bergman kernel proof" of Theorem 9.** Let $\psi_t := \psi_0 + u_t$. As will be shown below, differentiating $\mathcal{L}_{\omega_0}(u_t)$ gives

$$(6.7)\quad \partial_t\partial_{\bar{t}}\mathcal{L}_{\omega_0}(u_t) = \frac{1}{N}\sum_{i=1}^{N}(\|(\partial_t\partial_{\bar{t}}u_t)s_i\|^2_{\psi_t} - \|(\partial_{\bar{t}}u_t s_i) - \Pi_{u_t}(\partial_{\bar{t}}u_t s_i)\|^2_{\psi_t}),$$

where (s_i) is orthnormal wrt $\psi_t = \psi_0 + u_t$. Given this formula the argument proceeds exactly as in [13]; by the definition of Π_{u_t}, the second term inside the sum is the L^2- norm of the solution s to the inhomogenous $\bar{\partial}-$equation on X :

$$\bar{\partial_X}s = \bar{\partial_X}(\partial_t u))s_i,$$

which has minimal norm wrt $\|\cdot\|^2_{\psi_t}$. Now the Hörmander-Kodaira L^2-inequality for the solution gives

$$(6.8)\qquad i^{n^2}\int_X s\wedge\bar{s}e^{-\psi_t} \leq i^{n^2}\int_X \left|\bar{\partial_X}(\partial_t u_t)\right|^2_{\omega_{u_t}} s\wedge\bar{s}e^{-\psi_t},$$

using that $\omega_{u_t} > 0$. Hence, by formula 6.7,

$$\partial_t\partial_{\bar{t}}\mathcal{L}_{\omega_0}(u_t) \geq \frac{1}{N}\sum_{i=1}^{N}(\left\|(\partial_t\partial_{\bar{t}}u_t) - \left|\bar{\partial_X}(\partial_t u_t)\right|^2_{\omega_{u_t}})s_i\right\|^2_{\psi_t}$$

But since, by assumption, $(dd^cU + \pi_X^*\omega_0)^{n+1} \geq 0$ the rhs is non-negative (compare formula 3.25), which proves that $\mathcal{L}_{\omega_0}(u_t)$ is *convex* wrt real t. Note that $\mathcal{L}_{\omega_0}(u_t)$ is *affine* precisely when 6.8 is an *equality.* By examining the Bochner-Kodaira-Nakano-Hörmander *identity* implying the inequality 6.8 one sees that the remaining term appearing in the identity has to vanish. In turn, this is used to show that the vector field V_t defined by formula 3.6 has to be *holomorphic* on X (see [13]). Integrating V_t finally gives the existence of the automorphism S_1 in Theorem 9, as explained in section 3.1.

In [13] formula 6.7 was derived using the general formalism of holomorphic vector bundles and their curvature. We will next give an alternative "Bergman kernel proof". First formula 6.6 and Leibniz product rule give

$$\partial_t\partial_{\bar{t}}\mathcal{L}_{\omega_0}(u_t) = \int_X(\partial_t\partial_{\bar{t}}u_t)\beta_{u_t} + \frac{d\beta_{u_t}}{dt}_{t=0+}(\partial_t u_t)$$

Next, by formula 6.6 the second term may be expressed in terms of the Bergman kernel $K_t(x,y)$ associated to the weight ψ_t as

$$\frac{1}{N}\int_{X\times X}|K_t(x,y)|^2 e^{-(\psi_t(x)+\psi_t(y)}((\partial_t u_t)(x)(\partial_t u)(y) - \int_X \beta(\partial_t u_t)^2,$$

By simple and well-known identities for Toeplitz operators this last expression, for $t = 0$, is precisely the trace of the operator $T[\partial_t u_t])^2 - T[(\partial_t u_t)^2]$. All in all we obtain

$$\partial_t\partial_{\bar{t}}\mathcal{L}_{\omega_0}(u_t) = \frac{1}{N}\mathrm{Tr}(T[\partial_t\partial_{\bar{t}}u_t] + (T[\partial_t u_t])^2 - T[(\partial_t u_t)^2]),$$

for $t=0$. Expanding in terms of an orthonormal base s_i hence gives

$$\partial_t\partial_{\bar{t}}\mathcal{L}_{\omega_0}(u_t)=\frac{1}{N}\sum_{i=1}^{N}(\|(\partial_t\partial_{\bar{t}}u_t)s_i\|^2_{\psi_0+u_t}+\|\Pi_{u_t}(\partial_t u_t s_i)\|^2_{\psi_0+u_t}-\|\partial_t u_t s_i\|^2_{\psi_0+u_t}),$$

for $t=0$ (and hence for all t by symmetry) which finally proves 6.7, using "Pythagora's theorem".

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